International Symposium
Elementary Mathematics Teaching

Opportunities in Learning and Teaching
Elementary Mathematics

Charles University
Faculty of Education
Prague, the Czech Republic
August 18 – 22, 2019
International Symposium
Elementary Mathematics Teaching

Prague, the Czech Republic
Charles University, Faculty of Education

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Proceedings

Opportunities in Learning and Teaching
Elementary Mathematics

Edited by Jarmila Novotná and Hana Moraová

Prague 2019
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Since all papers and other presentations here are presented in English, which is not usually the
first language of the presenters, the responsibility for spelling and grammar lies with the
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Dear colleagues:

Welcome to SEMT 2019!

I am delighted to welcome you to the 15th biennial International Symposium on Elementary Mathematics Teaching [SEMT] at Charles University, Prague, Czech Republic, on August 18-22, 2019.

This is a landmark year for SEMT, being the 15th Symposium or 28 years since its inception. SEMT is now a well established international academic event with participants who are mathematics education researchers from all over the world. It is unique in its focus on the teaching of mathematics to children within the age-range of 5 to 12 years. Thus, SEMT provides an opportunity for participants to share ideas and experiences and to learn from and connect with researchers in elementary school mathematics education from around the world in a supportive, stimulating academic environment.

Organized by Charles University, under the leadership of professor Jarmila Novotná and Hana Moraová with the support of an International Program Committee and an International Advisory Board, SEMT offers participants a special opportunity to focus on mathematics education research at the elementary school level through a variety of excellent plenaries, research reports, posters and hands-on workshops. All research reports and posters are refereed to ensure a high quality for the scientific program. This year, the Symposium is framed by the theme, Opportunities in Learning and Teaching Elementary Mathematics, which is timely and important given the need for ongoing efforts to reform the teaching of mathematics to engage students meaningfully in learning and doing mathematics and help them to develop competencies needed for today’s world. The papers in SEMT 2019 Proceedings offer participants, and the mathematics education community in general, a variety of ways in which this theme has been and can be addressed.

The social activities, which are also important aspects to the SEMT experience, include a conference dinner and an educational excursion. In addition, being in historic, beautiful Prague is a definite bonus for those of us who do not live there! Whether it is your first visit or not, exploring and enjoying the city should be on your agenda.

I wish you all a memorable and inspiring SEMT 2019.

Olive Chapman
Chair
International Program Committee
Abstract

Opportunity to learn mathematics encompasses instructional activities that allow students to engage in and spend time “doing mathematics”. As part of such instructional activities teachers may also engage pupils to reason and communicate mathematically. Often tasks available in textbooks are not suitable for such activities. Over the last decade teachers in Singapore schools have been using several “What” strategies to engage pupils in reasoning and communication during mathematics lessons. This development was an outcome of a developmental research project that was carried out by the presenter from 2006-2008. This lecture introduces participants to some of the strategies and the accompanying classroom activities that have successfully engaged pupils in meaningful classroom discourse.

Keywords: opportunity to learn, reasoning, communication, elementary pupils, mathematics

Introduction

Opportunity to learn mathematics encompasses instructional activities that allow students to engage in and spend time “doing mathematics” (National Research

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As part of such instructional activities teachers may also engage pupils to reason and communicate mathematically. Often the intended curricula provides the direction for the learning of mathematics whilst textbooks play a more direct role in what is addressed in instruction (Wijaya, van den Heuvel-Panhuizen and Doorman, 2015). Kaur, Seah and Loh (2005) in their work with competent mathematics teachers, in Singapore, found that they were generally bound in their choice of learning tasks (tasks used by the teacher during instruction to develop a concept or demonstrate a skill or process) available in the textbook. A study conducted in the US by the American Institute for Research (AIR) found that textbooks in Singapore focused on practice exercises that emphasized procedural knowledge but lacked emphasis on reasoning and communication which facilitate higher order thinking skills (Ginsburg, Leinward, Anstrom and Pollock, 2005).

In 2006, the revised school mathematics curriculum for Singapore schools emphasised reasoning and communication as one of the processes. The findings of Kaur et al. (2005) and Ginsburg et al. (2005) led the author of this paper and her colleague Dr Yeap Ban Har to conceptualise a development project with the goal of equipping teachers with the know-how of tasks that are suitable for reasoning and communication. In the next section, the professional development project is briefly outlined.

Enhancing the Pedagogy of Mathematics Teachers – Reasoning and Communication (EPMT – R&C) project

Forty teachers, 22 from five secondary schools and 18 from five primary schools participated in the project. The PD was coherent with the needs of the teachers as the teachers relied heavily on textbooks for their daily work and there was a need for teachers to draw on textbook questions as starting points and craft tasks that would engage students in reasoning and communication. During the first phase of the PD, teachers attended training workshops conducted by the university professors. The workshops were organised as two modules, the first centred around crafting of tasks that would engage students in reasoning and communication and the second centred around teaching for understanding. Each workshop began with the university professor introducing the teachers to an idea. In the case of the first module, they were introduced to ideas of how typical textbook questions could be crafted into tasks that would engage students in reasoning and communication. The module introduced the teachers to eight strategies (Kaur, 2012; Kaur and Ghani, 2011). The second module focussed on the “why, what and how” of teaching for understanding. This module engaged teachers in planning lessons and crafting / selection of appropriate learning and practice tasks.

In the second phase of the PD teachers were encouraged to infuse in their lessons their learning from the training workshops they participated in during the first
phase of the project. Teachers were given specific assignments by the university professors. While teachers were working on their assignments, the university professors’ facilitated fortnightly meeting sessions during which teachers shared their work with the others and invited critique. It was during these sessions that teachers’ shared with the rest of the project participants their tasks, lessons (through video-records), students’ work and students’ voices. They invited both applause and critique. We must say that after the first few sessions, the activity picked up momentum and teachers became more “welcoming” of critique. It was during these sessions that teachers were meaningfully engaged in the production of pedagogical knowledge, creating and testing their plans, most importantly taking into consideration their students’ inputs like what made the lessons enjoyable and meaningful (Kaur, 2010; Kaur, 2013a; Kaur, 2013b). Using video records of their lessons they watched the performance of their students in class, reflected on their goals and evaluated their lessons. These actions led to revision/modification of plans for subsequent lessons. Towards the end of this phase teachers submitted their assignments. The assignments submitted by the teachers led to the publication of the resource *Pedagogy for engaged mathematics learning* (Yeap and Kaur 2010).

During the third phase teachers were left to work with their project mates in their schools to advance the knowledge they had gained from the first two phases. The university professors facilitated monthly meeting sessions during which project participants were engaged in a variety of activities:

i) They continued to share their “highs and lows” of lessons that engaged students in reasoning and communication and also lessons that “taught for understanding”.

ii) They prepared exemplars of mathematical tasks that were suitable for engaging students in reasoning and communication (primary and secondary), for publication as print resources (Kaur and Yeap 2009a, 2009b) for mathematics teachers in Singapore schools.

iii) Teachers participating in national conferences, school based and cluster level presentations prepared their presentations.

Following participation in the PD teachers from two schools went on to enlarge their community of practice and scaled up the intervention school-wide. The experts, teachers who had participated in the PD, were able to enlarge their school-based community of practice subsequently from four teachers each to 18 in the first school (primary) and 12 in the second school (secondary) (see Kaur, 2015 for details).

**The ‘What’ Strategies**

In the first phase of the project teachers were introduced to eight ‘what’ strategies for crafting tasks they could use in their lessons to engage students in reasoning and communication. In this section, we elaborate four of the strategies. The four
‘what’ strategies are: 1) what number makes sense? 2) what’s wrong? 3) what questions can you answer? 4) what’s the question if you know the answer?

**What number makes sense?**

In “What number makes sense?” pupils are presented with a mathematics version of a cloze passage, many pupils would be familiar with in their Language lessons. Pupils are presented with problem situations from which numerical data is missing. A set of numbers is provided and pupils determine where to place each number so the situation makes sense. The steps given as part of the problem sheet help to focus the pupils on the steps they need to take and also explain their thinking. The teacher must ensure that group interaction followed by class discussion occurs so that pupils have the opportunity to explain their thinking and also learn of ways of solving problems that differ from their own. As pupils work through tasks of this nature, they practice computation and increase their repertoire of problem-solving skills. Reasoning skills are improved by being exposed to a variety of ways to solve a problem (Krulik and Rudnick, 2001). Such a task can be easily crafted from a typical textbook question. Figure 1 shows a typical textbook question and a what number makes sense task based on the textbook question.

**What’s wrong?**

In, “What’s wrong?” the pupils are provided with an opportunity to use their critical thinking skills. They are presented with a problem and its solution. However the solution contains an error, either conceptual or computational. The pupil’s task is to discover the error, correct it and then explain what was wrong, why it was wrong and what was done to correct the error (Krulik and Rudnick, 1999). The teacher must ensure that pupils are engaged in class discussion after completing the task either in small groups or individually so that they hear ways of solving problems that differ from their own Furthermore the group interaction that occurs during these discussions often leads to deeper mathematical understanding (Krulik and Rudnick, 2001). Such tasks are not difficult for teachers to craft as they are constantly exposed to such errors pupils make in class and in their written assignments. Figure 2 shows one such task.

**What questions can you answer?**

In “What questions can you answer?” kind of tasks pupils are provided with situations that include numerical data and / or geometrical figures and are asked to generate questions that can be answered using the given information. This activity is both creative (as pupils have to pose more than one question and hence stretch beyond the obvious) and critical (as pupils have to make sure that the questions they pose are solvable). The teacher must ensure that after completing the task pupils show case their questions together with solutions and engage in class discussion so that they realize the breadth and depth of questions that can be
constructed with the information. The sophistication of the questions posed by individual pupils show their developmental level and this is excellent feedback for the teacher. Figure 3 shows one such task.

**What’s the question if you know the answer?**

In “What’s the question if you know the answer?” kind of tasks pupils are presented with the context and data of a problem but with the question/s missing. They are given a solution and asked to write a question that matches it. Such tasks provide an opportunity for pupils to engage in critical thinking skills. Whole class discussion must precede individuals working on the task as it is important for pupils to recognize that there may be several questions that have the same answer. Figure 4 shows one such task.

**Textbook question**
A box contained 42 apples. 12 of them are green and the rest were red.
Find the ratio of the number of green apples to the number of red apples.

What number makes sense?
Read the problem. Look at the numbers in the box.
Put the numbers in the blanks where you think they fit best.
Read the problem again, do the numbers make sense?

| 2 | 5 | 12 | 30 | 42 |

**Apples in a box**
Mary bought a box of red and green apples.
The box has _____ apples. There are more red apples than green apples.
There are _____ red apples and _____ green apples.
The ratio of the red to the green apples is _____ : _____.

Figure 1: What number makes sense task (Kaur and Yeap, 2009, p. 14)

**Caili saved $6.85**
She saved $1.25 more than her sister.
How much did the two girls save in all?

Susan’s solution:  $6.85 + $1.25 = $8.10
$8.10 + $6.85 = $14.95
The two girls saved $14.95 in all.

There is something wrong with Susan’s solution?
1. Show how you would find the answer to the problem.
2. Explain the mistake in Susan’s solution.

Figure 2: What’s wrong task (Kaur and Yeap, 2009, p. 25)

The menu for a Pizza Hut is as follows:

<table>
<thead>
<tr>
<th>Menu Item</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rino’s Pizza Hut</td>
<td></td>
</tr>
<tr>
<td>Large Pizza</td>
<td>$21.50</td>
</tr>
<tr>
<td>Regular Pizza</td>
<td>$15.50</td>
</tr>
<tr>
<td>Small Pizza</td>
<td>$ 7.00</td>
</tr>
<tr>
<td>Spaghetti</td>
<td>$ 9.50</td>
</tr>
<tr>
<td>Garlic Bread</td>
<td>$ 2.50</td>
</tr>
</tbody>
</table>

Write two questions with the above information.

1. Question 1

2. Question 2

Find answers to your questions. Show all your workings.

Figure 3: What questions can you answer task (Kaur and Yeap, 2009, p. 40)

Madam Lee has a bale of cloth measuring 150 m in her shop. A customer asked her to sew 10 cushion covers. To make a cushion cover she uses 3 m of cloth. She sold 21 m of the cloth to another customer.

1. What’s the question if the answer is 30 m?
2. What’s the question if the answer is 99 m?
3. What’s the question if the answer is 33 ?
4. What’s the question if the answer is 1/5 ?

Figure 4: What’s the question if you know the answer task (Kaur and Yeap, 2009, p. 64)
Enactment of a reasoning and communication lesson by a grade one teacher in the project

In this section we present a grade one mathematics lesson during which the teacher used the strategy “What’s the question if you know the answer?” and engaged her pupils to reason and communicate their reasoning during the lesson.

Objectives of the tasks created by the teacher

Figures 5 and 6 show the two tasks that the teacher created and enacted in her lesson for grade one pupils. The objectives provided by the teacher for the use of tasks X1 and X2 in her lesson were

i) to provide students with an opportunity to reason and communicate,

ii) to review numbers less than 40 and the four operations, +, −, × and ÷.

### Eggs-cellent?

Mrs Lee has a farm. She collects eggs from her farm every day.

<table>
<thead>
<tr>
<th>Day</th>
<th>Number of eggs collected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday</td>
<td>23</td>
</tr>
<tr>
<td>Tuesday</td>
<td>17</td>
</tr>
<tr>
<td>Wednesday</td>
<td>8</td>
</tr>
<tr>
<td>Thursday</td>
<td>5</td>
</tr>
<tr>
<td>Friday</td>
<td>10</td>
</tr>
</tbody>
</table>

What’s the question if the answer is 5?
What’s the question if the answer is 25?
What’s the question if the answer is 6?
What’s the question if the answer is 30?

Figure 5: Task X1

### Old MacDonald had a Farm

There are 5 cows and 12 ducks at a farm.
Each cow has 4 legs and each duck has 2 legs.

What is the question if the answer is 17?
What is the question if the answer is 7?
What is the question if the answer is 20?
What is the question if the answer is 44?

Figure 6: Task X2
How the tasks were enacted

This 60 minutes lesson was a review of numbers less than 40 and the four operations (addition, subtraction, multiplication and division). The lesson began with the teacher reviewing number sentences using numbers less than 40 and the four operations. Next the teacher engaged the whole class step by step in doing task X1. The students sat on the floor at the front of the whiteboard in the class which displayed task X1. The teacher explained to the students what the task was all about and very carefully led them to craft questions that resulted in the answers given. This was followed by seatwork that adopted the Think-Pair-Share strategy. Students worked in pairs on task X2, while the teacher walked around the class providing guidance. Teacher also gathered common difficulties the students faced and drew the attention of the whole class to them and scaffold their thinking. Teacher also monitored the progress of students’ work and facilitated the sharing amongst the students in groups. As there was insufficient time for the students to work through all the items in task X2, the teacher limited classwork to only the first two. 50 minutes into the lesson, the teacher drew the attention of the whole class to the diversity of questions the students had crafted and also some very creative ones. The lesson ended with the teacher assigning the students to do the last two items of task X2 as homework.

Teacher’s self-evaluation of her lesson

The teacher was guided in the self-evaluation of her lesson by the following prompts:

i) Was this lesson different from lessons in the past that you conducted? If so, what enabled you to make it different?

Teacher’s response: In the past, I reviewed numbers less than 40 and the four operations by doing whole class oral work and individual written work. My students did exercises like: $7 + 9 = ?, 12 - 5 = ?, 5 \times 4 = ?, 10 \div 2 = ?$ They also made number sentences with three numbers, like 7, 8, 15. This lesson is different. I used the strategy “what is the question if the answer is?” that I learned in the project. I want my students to make number sentences, but only one number is given, the answer. They must reason and pick the other two numbers from the data provided. I want them to discuss with their classmates. I also want them to see many possibilities their friends may have different number sentences. I want them to articulate their questions. They always read questions in their textbooks. This is their chance to put the questions together. The project has introduced me to many good strategies and I like to use as many of them as possible in my lessons.

ii) What were the challenges you faced in executing the lesson plan?

Teacher’s response: The biggest challenge was to maintain the cognitive demand of the task, i.e. not to take over the thinking that I wanted my students to do. So,
for the whole lesson, I answered all the queries of my students with appropriate counter questions, such as “what do you think?”, “how would you check”, etc. I also over planned my lesson. There was not enough time towards the end of the lesson for presenting all the diverse and creative questions the students had formed.

iii) Would you use the strategy again in future lessons?

Teacher’s response: My students enjoyed the lesson very much. They got lots of opportunity to talk. The diverse responses of their classmates introduced them to look for more than one way. I will use this strategy again in my lessons but the purpose of my lesson will decide when and for what topic I may use it. I personally find this strategy useful for review lessons.

Resources created by teachers in the project for teachers in Singapore schools

Mathematical tasks created by the teachers in the project and lessons they enacted using such tasks were collected at an on-line site shared by all the participants. During the last phase of the project these resources were put together for publication (see Figure 7). Two print resources were the outcome. The first was *Pathways to reasoning and communication in the primary school mathematics classrooms* (Kaur and Yeap, 2009a). It explains the why what and how of reasoning in the mathematics classrooms and introduces each of the eight “what strategies” and shows examples created by teachers in the project. The eight ‘what’ strategies are 1) what number makes sense? 2) what’s wrong? 3) what would you do? 4) what questions can you answer? 5) what’s missing? 6) what if? 7) what’s the question if you know the answer? and 8) what’s the question?. The second is *Pedagogy for engaged learning* (Yeap and Kaur, 2010). It includes pedagogies that were engaging in the classrooms. Each pedagogical approach is illustrated with examples of mathematics lessons from the project. The approaches are 1) learning by inquiry, 2) learning by doing 3) learning by interacting, and 4) learning by reflecting. Each chapter delineating an approach begins with a description of the approach, followed by examples of lessons using the approach. It ends with professional development tasks for further work and study by individuals or groups. Other than showing work of the teachers in the project the two objectives of the resources were to 1) create awareness amongst teachers about pedagogies that engage pupils during math lessons, 2) stimulate school based professional development, and 3) provide stimulus for teachers to engage in research and publication.

The project was completed in 2008, and the print resources distributed by the Ministry of Education to all schools in Singapore by 2010. These resources have fueled school-based professional development until the present. In some of the primary schools in Singapore, the ‘what’ strategies are an integral part of
mathematics instruction that facilitates student-centred learning and helps learners make their thinking visible.

Figure 7: Resources created by teachers in the project

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**HOW TEACHING MATHEMATICS IN ELEMENTARY SCHOOL CAN SUPPORT FINANCIAL EDUCATION**

*Annie Savard ©*

**Abstract**

This communication aims to explore the importance of Mathematics in Financial Education. The emergence of the Financial Education field is briefly presented before portraying its epistemological intersection with mathematics. This intersection is

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highlighted by the two different meanings of measurement: representational and pragmatic. These two meanings are present in different mathematical tasks. Three examples of tasks are given, in which cognitive and social aspects are described in order to study some financial practices in different contexts. For each example, the kind of citizenship will be exposed and discussed.

Keywords: mathematics, financial education, measurement, citizenship

Introductory context

Introduction

Since the last decade, Financial Literacy has received a lot of attention all around the world. Since 2012, the OECD tests Financial Literacy among 15-year-old students through PISA assessment. Many countries, such as Australia, Canada, Japan, France and United States, have incorporated Financial Literacy in their curriculum in various ways; sometimes as a separate course, other times incorporated in mathematics or social studies, or even as a cross-disciplinary field.

Teaching Financial Literacy is not only a major international trend in middle and high school, it is also a major trend in elementary school. One important aspect to consider is that elementary school provides the opportunity to approach the topic from an interdisciplinary perspective, which is fundamentally different from high school. However, if it is officially written in many curricula, little is said on how to teach it, especially in an interdisciplinary way. In this regard, mathematics offers essential and authentic opportunities to develop an understanding of financial concepts.

In this communication, I propose an exploration of the importance of Mathematics in Financial Education by presenting some results coming from different research projects I have been conducting in this area.

From Financial Literacy to Financial Education

Financial Literacy is the terminology used to talk about the ability to mobilize knowledge and skills in real-life financial situations. It is the desired outcomes of educational programs or curricula. This terminology is used, among others, by the Organization for Economic Cooperation and Development (OECD) to talk about their financial literacy assessment that is included since 2012 into their triennial Program for International Student Assessment (PISA). This assessment examined the financial literacy of 15-year-olds students. The OECD (2016) defines Financial Literacy as:

Financial literacy is knowledge and understanding of financial concepts and risks, and the skills, motivation and confidence to apply such knowledge and understanding in order to make effective decisions across a range of financial contexts, to improve the financial well-being of individuals and society, and to enable participation in economic life (p. 85).
The financial knowledge is presented into four content areas:

1. Money and transaction;
2. Planning and managing finances;
3. Risk and reward;

As the desired outcomes of educational programs, it makes sense to assess Financial Literacy. Now that we know the desired result of teaching financial concepts it is appropriate to conceptualise the field of Financial Education.

**Financial Education and Mathematics Education: an intersection**

Financial Education is an emergent field, especially in Education. Very few Education scholars investigated this field in the past or in present time. Most of the researchers are coming from other fields such as anthropology, economics, finance or sociology. In my opinion, it is time that scholars in Education take the lead of this field, especially researchers in Mathematics Education. The reasons are coming from the epistemology of Mathematics and Finances: for decades, the concept of money is part of elementary and secondary school curricula from all around the world. Money is thus conceptualized as a measurement concept. Money is then a unit of measurement that allows for the comparison of attributes to objects and their relationships to numbers. This meaning of measurement is named representational measurement (Hand, 2016). In textbooks, elementary students will be asked to know the name of the currency, the value of the bills and coins, and perform different arithmetic tasks in word problems about money. Those word problems use money as a financial context as a rationale to perform operations and connect mathematics to real-life situations. For decades, financial contexts are presented in elementary and secondary mathematical textbooks to solve mathematics problems. Even the OECD, through PISA, proposed financial contexts to assess mathematical literacy among 15 years old students (Savard, 2018).

Another meaning of measurement highlights the complexity to measure objects or phenomenon subjectively, namely the pragmatic measurement. The pragmatic measurement takes into consideration many attributes of an object or a phenomenon to provide a value and an indication on how measurement will be performed (Hand, 2016). Speaking about money, the representational measurement is mainly a mathematical epistemology, where the pragmatic measurement is mainly a financial epistemology (Savard, Cavalcante, Turineck and Javaherpour, submitted; Caprioara, Savard and Cavalcante, in press).

**A Framework for Teaching Financial Education**

Financial Education is the field of teaching and learning the financial dimension of the production and management of resources. Cognitive and social aspects are
considered, such as financial practices in different contexts. Producing resources means developing actual and new resources, which includes entrepreneurship education. Managing resources means managing individual and collective resources in order to develop well-being for oneself and their communities. Teaching Financial Education should aim to develop cognitive understanding of sociocultural practices about producing and managing individual and collective resources in order to engage all students in developing citizenship competencies. Thus, teaching Financial Education in mathematics classrooms should start by creating a learning environment where students should study an object or a phenomenon coming from a sociocultural context involving finances. This sociocultural context could be a context where financial practices are pictured, such as buying things in a store, or it might be a context where financial concepts are present such as profit or compound interests. This sociocultural context should be presented into a learning situation in which mathematics is a privileged tool to develop a cognitive understanding. For instance, mathematics might be used to modelized the situation and make sense of it. Thinking rationally about the object or the phenomenon studied using mathematics supports the development of a cognitive understanding. Consequently, the knowledge and understanding developed in the cognitive context are about finance and mathematics. Which is not enough, because we want students to engage critically (Lipman, 2003; Paul and Elder, 2001) in those financial practices and make the best decisions for them and their community (Caprioara, Savard and Cavalcante, in press; Savard, 2017). This is why it is important to use mathematics to highlight the citizenship context that might be explicitly or implicitly presented into the situation. For instance, the situation on buying things in a store might lead to questions about fair trade, while the situation on profit might lead to discuss sustainability. The situation on compound interest might lead to save money for education. Discussing those financial practices engages students in citizenship, which is the participation by a member of a community of social practices in a critical and responsible way to the practice in question. This is a democratic way to the evolution of the social practices and thus the community itself (Ten Dam and Volman, 2004). In this point of view, critical thinking and decision making are considered as citizenship competencies (Savard, 2008). According to Westheimer and Kahne (2004), there are three kinds of citizens: the personally responsible citizen, the participatory citizen, and the justice-oriented citizen. The personally responsible citizen acts in a responsible manner in his/her community, while the participatory citizen acts actively in the community at the local, state, or national level. The justice-oriented citizen analyzes and questions issues about social justice. For instance, the responsible citizen might donate money or food to a local organization for charity, the participatory citizen volunteers at the fundraising activity, and the justice-oriented citizen will try to find a way to eradicate poverty.
Three illustration of teaching mathematics and financial concepts

The draw

I conducted a study among twenty-seven Grade 4 elementary school students. I wanted to investigate their probabilistic reasoning. I developed a teaching experiment (McClain, 2002) composed of six learning situations that were designed for each type of probability: subjective, theoretical and frequentional (experimental). At the very end of the experiment, few months after the last situation was presented, I proposed to students three fictional vignettes on a daily life situation involving probability. Each vignette was presented on different days. Students were first asked to respond individually on a sheet. The teacher/researcher collected them and facilitated a whole-class discussion on the vignette. After the discussion, students were again asked to write down what their answer to the question would be after the discussion on a new sheet.

The first vignette that was presented was about a fundraising activity for a charity. Students should tell the reasons to participate or not using mathematics and financial concepts:

A local youth group from your neighbourhood organize a draw to raise money for an organization. They sell 100 raffle tickets at $1 per ticket. A big prize of $25 is drawn. They asked you to buy a ticket with your money.

What do you do? Why?

How much profit will the local youth make?

This vignette has two cognitive contexts: mathematics and financial. The cognitive context on mathematics is about probability. Students should make a model using theoretical probability in order to decide if they will participate or not. In addition to that, they have to define the profit they might make if they choose to participate. The cognitive context on finances is about the money they will make or not, i.e. the profit or the loss. The loss could be seeing as a financial risk (Savard, 2015).

The sociocultural context is about an internationally well-known financial practice, more specifically on designing, selling and buying lottery tickets, which is a gambling activity. Usually, lottery tickets are sold as a cheap price and provide opportunity to receive more money in exchange. Most of the time, the probability to win is pretty low.

The citizenship context is about organizing a draw for an organization. It has two different kinds of citizens: participatory and responsible. The vignette highlights a participatory citizen practice by staging a local youth group who organizes a fundraising activity. The task can put the students in a position of being personally responsible citizen, because they were asked to support the fundraising activity.
The different answers provided by students highlighted some aspects of the contexts. The aspects of the cognitive context that were related to mathematics was raised by some the students to make the decision to buy or not a lottery ticket. Thus, some students looked at the probabilities to win:

_**Mia:** _Well, I say no because even if it's just for $1, then, well you've don’t really have too much chances of winning if you buy a ticket. You have one chance in 100 of winning (V1.182).

She estimated that her probabilities of winning were low. She took into account that all the tickets would be sold to show an awareness of the concept of the relative theoretical probabilistic structures, or 1 chance in 100. Still in the cognitive context, another student compared the possible benefits of buying a ticket, in other words, the profit she will make. She based her rationale on the quantification of the gain, which is mathematics. In her case, she did mention the probability of winning:

_**Magalie:** _I would say no, because first, they will have the triple in their pockets. Then, he said that just $1, but maybe you have ... basically, is that you do not earn $25. You win, you win $24. That is like a catch because you spent $1 and won $25. At the end, you won $24 (V1. 182).

She addressed the expected gain, which is a financial concept. She addressed it in a critical thinker way by considering the fact that $1 have to be subtracted from the amount won, because it is the money invested. In fact, the prize of $25 equals a profit of $24. Which it is not obvious at a first glance. She questioned the real value of the prize, which is a sign of critical thinking (Lipman, 2003). Rational thinking should be performed to go over the $25 prize. She also questioned the money raised by the youth group. In her case, thinking about the money led her into the sociocultural context, where she considered that the profit made by the youth group, was triple of the prize. Money is here used to measure the gain or profit made by the student and the youth group. In this sense, the mathematical meaning given to money is a representational measurement. Magalie was quite skeptical about the use of the money by them. She was not alone. The citizenship context was very important for most of the students, where they showed some critical thinker skills. During the discussion, right after writing the vignette for the first time, they were very concerned about the youth group and their will to raise money. They said that they did not know where the profits raised would go, and it was the central argument in their decision-making. In fact, they were afraid that the youth group will keep the money for themselves instead of giving it to the organization. They were also concern about the organization: what kind of charity it was and for what reason they need the money. They questioned the integrity of the organization as well as the validity of their ethical action, as they assessed the charity organism before deciding if the ethics was important enough to invest money. For them, where the money went was more important than the money to
be won. They did not care about winning or losing money in this case. The citizenship context used all the space and obscured the mathematical context.

As the teacher of these students and also the research of the study, I proposed to the students to consider that the youth group is serious, and we can trust them. I also had to name a charity organization that all students in the classroom knew about: a charity organization that raise money for children in hospital. Only then, the majority of the students can pay a different attention to the vignette. Among students, 16 of them wanted to support the charity organization. They based their decision on the ethics of the situation instead of their personal comfort. Some even said that it was a donation to charity and that the gain was not important for them. Another ethical aspect raised by a student was about the legal age to buy lottery tickets:

Marco: A lottery, is all the time 18 years old and over. Well, we were not supposed to buy it, that I would say no there (V1.202).

The student questioned his participation from a legal point of view. In a sense, this shows a justice-oriented citizen thought, where an analytical point of view questions the practice.

The inappropriate price

Another example is coming from the work I did with my colleague Daniela Caprioara and my PhD student Alexandre Cavalcante on analyzing mathematical and financial tasks in Romanian mathematical textbooks. The tasks required students to estimate the price of some objects. Figure 1 shows the task on estimating prices:

![Translation: Reproduce the table below and list the price you think is inappropriate: (pencil, pen, sneakers, training, balloon, cake, bread).](image)

Figure 1: Tasks on estimating prices
This task has two cognitive contexts: mathematics and financial. The cognitive context on mathematics is about arithmetic magnitude. It is possible to look at the numbers and find the biggest different number for each item. For example, it is easy to see a big difference between 1, 85 and 2. The cognitive context on finance is about representational measurement. It is possible to think about the effective price and tell which one doesn’t make sense. For example, it is possible for a student to say that he or she already bought a pencil and it is around 1 or 2 lei. The cognitive context on finance is also about pragmatic measurement. It is possible to assess the value of the items by comparing the price with another item or by the value of the amount of money involved. For example, a pencil cost less than sneakers. The price of sneakers is usually high, because it is a pair of shoes. Asking students a justification about their answer might highlight their reasoning and create an opportunity de develop about the sociocultural and the citizenship contexts.

The sociocultural context is about the social practice of buying and selling them for the amount of money asked, the prices. The currency used is also an element of this context. Each item could also be discussed. For example, what those items are needed for.

The citizenship context is about questioning the price of the item and make a decision about buying an item that price. For example, how to fix prices opens a door about producing and selling goods and services. It is about cost of production and making profit when selling them. It is also about who produced the item and who is selling them. This might lead to discuss fair-trading, sustainability and child labour. This kind of citizenship might be explored with students. The personally responsible citizen might buy local products, the participatory citizen might inform other students about fair-trading, and the justice-oriented citizen might question supermarket about some of their products.

Making decisions about buying an item could refers to a need to use it for writing (pencil) or eating (bread), or to shows a social status (sneakers). For instance, why someone might be willing to pay a pay a big amount of money for an item that it is available cheaper might be a strong discussion to develop critical thinking that supports decision making (Savard, 2017). Again, the choice made might highlight different kinds of citizenship. The personally responsible citizen might decide to pay more to support the local producers within his/her community, the participatory citizen might decide to organize an event to support local producers, and the justice-oriented citizen might create a petition to have local products sold in departments store chains or in supermarkets.

The piggybank enigma

The last example is coming from my work with my colleague Elena Polotskaia on task design, more specifically on Mathematically incoherent situation (MIS) (Savard and Polotskaia, 2017). That kind of tasks foster the reasoning by
questioning the problem. The task doesn’t have question written in the text: the numbers together just don’t work. Searching what is wrong lead students to represent the relationships structures in the text and discuss them. Another task is to associate the representation and the discussion. I present a translated text and the first discussion coming from the book authored by Polotskaia, Gervais and Savard (2019):

Brothers Nicolas and Patrick received a piggybank.

Discussion 1

Nicolas: I put 7 dollars in our piggybank.

Patrick: I added some dollars me too.

Mother: You have 15 dollars in your piggybank.

This task has two cognitive contexts: mathematics and financial. The cognitive context on mathematics is about arithmetic, more specifically on additive relationships (Polotskaia and Savard, 2018). The cognitive context on finances is about the money saved from a representational measurement point of view, because it is about measuring the amount of money saved.

The sociocultural context is about the social practice of saving money in a piggybank. This practice of using a piggybank is well-known for now, but it might not be the case in a near future. There is a growing international trend of not using physical money anymore, but instead using digital money. However, saving money is still possible using digital money. Some apps are specialized in that. Saving money as a social practice is an important aspect of finance and might lead to discuss debt, purchase power and investment.

The citizenship context is about putting money together in a collaborative way and about why saving money. It is about questioning the practice and looking at the reasons to save money. For instance, students might adopt the point of view of a personally responsible citizen by saving for buying a game, the point of view of a participatory citizen by saving for later deposit in a bank account, or adopt the point of view of a justice-oriented citizen by questioning why some children don’t have the opportunity to save money.

Concluding remarks

Those three examples show three different financial practices: buying lottery tickets for a charity, estimating prices of goods, and saving money in a piggy bank. All of these financial practices might be performed in daily life by students. Therefore, the development of a cognitive understanding about them is necessary to provide them the rationale for making critical decision in their life and thus develop citizenship.

The role played by mathematics in Financial Education is more than accessory: it is key to make sense of the practice. It is like in Physics: without mathematics, it
is impossible to explore deeper the field. Furthermore, the epistemology of Mathematics and Finance share a strong intersection, influenced by the two meanings of measurement: representational measurement and pragmatic measurement. Both are necessary to fully develop a cognitive understanding of mathematics and finance.

References


ELEMENTARY CONCEPTUAL PROGRESSIONS: REALITY CHECK + IMPLICATIONS

Ron Tzur

Abstract

I articulate a conceptual progression of students’ elementary mathematics—consisting of pre-numerical schemes, numerical schemes, schemes for multiplicative reasoning, and schemes for fractional reasoning. I point out concepts in elementary mathematics these schemes underlie. Then, I provide data to support benefits of this progression for practice and theory building. I discuss implications for mathematics teaching and teacher development, including (a) availability of those schemes in teachers’ mathematics and (b) the need to shift from typical approaches to a student-adaptive pedagogy.

Keywords: conceptual progression, number, multiplicative, fractions, constructivism

An essential question asked of us, researchers and teacher educators in mathematics education, is: What role can research play in practice? This is a key challenge for a profession that sets as its chief goal to promote mathematics learning and teaching (Simon et al., 2018). The conceptual progression I present to address this question summarizes a constructivist research program established by Les Steffe and his colleagues (Steffe and Cobb, 1988; Steffe and Olive, 2010). I begin with schemes that give rise to a child’s concept of number (i.e., pre-numerical schemes). Then, I articulate the foundational concept of number as a composite unit—a unit the mind produces by putting together smaller units (e.g., 5 is a unit made of five 1s, or 3 and 2). This leads to 6 schemes in multiplicative and 8 schemes in fractional reasoning.

A Progression Toward Number (Pre-Numerical)

Table 1 presents a progression of six pre-numerical schemes. For each, I present the term, the key units/operations it refers to, and an example to illustrate it (T stands for teacher, C for child; numerals stand for words to save space).

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<thead>
<tr>
<th>Scheme</th>
<th>Units/Operations</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rote Counting</td>
<td>Utter number words one-by-one in sequence.</td>
<td>T: How far can you count? C: 1-2-3 ... 18-19-20-21 ...</td>
</tr>
<tr>
<td>One-to-One Correspondence</td>
<td>Coordinate number words with items.</td>
<td>T: How many cubes are there? C: (Points to each cube), 1-2-3-... 6-7 cubes.</td>
</tr>
<tr>
<td></td>
<td>b) “Pair” items to figure out which has more.</td>
<td>T: What is more, 7 cubes or 4 cubes? C: (Pairs up 4 cubes, sees extra): 7 is more.</td>
</tr>
<tr>
<td>Manyness, with Subitizing as an early onset of it</td>
<td>Manyness say last number word without recount.</td>
<td>T: How many fingers are there? C: (Subitizes first hand) Five; then, 6-7.</td>
</tr>
<tr>
<td></td>
<td>C: The same: 7.</td>
<td></td>
</tr>
<tr>
<td>More/Fewer</td>
<td>Count for manyness; T: Take 7 cubes for you and give me 4. Who</td>
<td></td>
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</tbody>
</table>

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compare two collections has more and by how many?

of tangible items

C1: (Counts items, pairs up) I do. 3 more.

C2: I have more; I have 7. So, I have 7 more.

**Figurative Counters**

Count tangible items other than those asked for; anticipate manyness.

T: (Hides 4 cubes) How many more?

C1: Peaks under the cover (still perceptual).

C2: (Counts up 4 fingers; raises a finger per hidden cube) 5-6-7; I have 3 more.

**Counting-All**

Start at 1; Count 1s; find manyness of two collections.

T: How many are 7 cubes and 4 cubes in all?

C: (Counts the 7 cubes, then continues with the 4) 1-2-3 … 6-7; 8-9-10-11.

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Table 1: Six schemes for pre-numerical reasoning.

I stress four critical points about the pre-numerical schemes. First, starting at rote counting and throughout, a child’s use of number words and numerals should not mislead a teacher to attribute a concept of number to the child. The reason – in all six schemes the child operates on 1s, not on units (numbers) larger than 1. Second, in the More/Fewer scheme, C2 illustrates a common error found in children—incorrectly linking a conventional term to one’s mental actions. Because the term (more/fewer) is a convention, a teacher should point out the acceptable referent. Third, there are two types of one-to-one correspondence: (a) linking number words with items and (b) pairing items to compare collection sizes. Often, the second type is overlooked, which explains erroneously applying more/fewer (C2 above). Fourth, both manyness and counting-all involve operating on 1s and counting from 1; they differ in the child’s intention (count-all to add 1s in two collections).

**A Progression in Numerical Reasoning (Whole Numbers)**

With counting-on as an indication of the child’s concept of number, I present six schemes (Table 2) in a progression of numerical reasoning that underlie addition and subtraction. The child’s shift from operating on singletons (1s) to operating on numbers as composite units is a formidable conceptual leap. Understanding the nature of this change and how it may come about over a long period of time, and appreciating the non-trivial challenge it presents, seem vital (for more, see Fuson, 1992; Tzur and Lambert, 2011). Yet, adults who already have a concept of number may diminish this leap to a trivial extension of counting-all into counting-on.

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<table>
<thead>
<tr>
<th>Scheme</th>
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<tbody>
<tr>
<td><strong>Counting-On</strong></td>
<td>To add two numbers, regard T: You have 8 cubes; I give you 7 more. one addend as a “single entity” to start from; count 1s of the other unit, keeping track to stop at its last item. C1: (Uses fingers) 8; 9-10-… 14-15.</td>
<td>C2: (Abstract) 8; 1-is-9, 2-is-10, … 7-is-15. C3: 8; 9- … 15-16-17 … Oops; didn’t stop.</td>
</tr>
</tbody>
</table>
Counting-up-to
To find how items needed to reach a target, start from the given addend, operate on 1s while keeping track to stop at the target total.

T: You have 8 cubes; How many more cubes do you need in order to have 15?
C1: (Same operations as in counting-on; stops at 15; looks at fingers; sees one full hand & 2 more): 7 more.
C2: 8; then 9-is-1, 10-is-2, … 15-is-7.

Doubling (often used for n & n+1)
To add numbers, decompose one number, so it includes a unit equal to the other; double it; compensate.

T: You have 8 cubes; I give you 7 more. How many cubes would you have in all?
C1: 8 is 7+1. 7+7 is 14. One more is 15.
C2: 8+8 is 16. Take the 1 from 16. So, 15.

Break-Apart-Make-Ten (BAMT); a “through-decade” scheme
To add numbers, decompose one addend into smaller units ( >1), one to compose the other addend into 10; recompose with the other, small unit(s).

T: (Same task for 8+7.)
C1: I know 8+2 is 10. I give 2 from 7 to 8 to make 10. Then add 5. I’ll have 15 cubes.
C2: 8 is 5+3; 7 is 5+2; add 5+5+(3+2)=15.
C3: Give 2 to 28 to make 30; add 5. So, 35.

Counting-back-from (Could turn into reverse-BAMT)
To subtract numbers, start at the larger number and count 1s in reverse order, while keeping track to stop at the other number’s last item.

T: You have 15 and I have 8. How many more cubes do you have?
C1: (Possibly using fingers) I start with 15. Minus 1-is-14, 2-is-13, … 6-is-9, 7-is-8.
C2: Break 7 into 5 and 2. 15 minus 5 is 10. Then, minus 2 more is 8 cubes.

Relating Partial Composite Units (“Fact-families”) To find sums/differences, anticipate result of operating on pairs of composite units, conceiving of the total as two smaller units: whole = part + part.

T: What can you say about 8, 7 and 15?
C1: 15 is 8+7.
C2: 15 is also 7+8. Order does not matter.
C3: If you have 15 cubes and I have 8, I need 7 more cubes to be equal (15-8=7).
C4: I also see 15 as 8+(2+5).

Table 2: Six schemes for numerical reasoning

Constructing number as a composite unit – an abstract, single unit composed of 1s or other smaller units that is symbolized by a word/numeral – is a conceptual keystone of the child’s mathematics. A child can meaningfully add or subtract precisely because a mental image of number as a composite unit allows her to compose and decompose it. Moreover, having a mental image supporting anticipation of, say, 15 being composed of 8 items and 7 items, or 10 and 5, or 13 and 2, etc., can give rise to the child’s consideration of such compositions as equivalent (hence, also commutativity). The conceptual power added to the child’s reasoning by constructing number as a composite unit is a major reason why, in this progression, we choose counting-on as a first indicator of a concept of number, whereas counting-all, which the child uses to add, is considered pre-numerical.

Research has repeatedly pointed to counting-on, doubling, and BAMT as different methods children may use to correctly solve addition problems, such as 8+7
(Fuson, 1992). We found a statistically significant difference in multiplicative reasoning based on the additive strategy a child used spontaneously (Tzur et al., 2017; see next section). We infer that when a child uses counting-on, she takes for granted and start from a given addend as a composite unit while operating on 1s that constitute the other addend. Doubling indicates a stronger concept, as the child operates on both addends as composite units, while using a unit of 1 to decompose one addend or the sum (e.g., 8-1 or 16-1). BAMT indicates a stronger concept than doubling, as the child decomposes a few numbers into sub-units (all larger than 1).

A Progression in Multiplicative Reasoning (Whole Numbers)

Having constructed number as a composite unit that one can decompose at will supports constructing schemes for operating multiplicatively on such units. At the heart of such reasoning is the 1-for-many coordination of units (Clark and Kamii, 1996), while applying counting operations to composite units (Steffe, 1992). For example, to figure out the total of candies needed to fill 4 bags with 3 candies each, a child may use fingers, or numbers, to count: 1-bag-is-3-candies, 2-is-6, 3-is-9, 4-is-12. Such coordination portrays the shift from additive to multiplicative reasoning as yet another conceptual leap, not only because the child counts two different sequences of numbers (e.g., 1-2-3-4 coordinated with 3-6-9-12) but also because the child distributes items of one composite unit (e.g., 3 candies) over items of another composite unit (e.g., 4 bags) that produces a different unit (12 single candies). The goal for children’s learning is to conceive of the product as both a composite unit made of 1s and as a unit composed of units of units (e.g., 12 as a unit of 4 units of 3 singletons). Simon Kara, Norton and Placa (2018) articulated multiplication in a way that addresses this conceptualization. The first scheme in Table 3, called multiplicative double counting (mDC) alludes to it. Following Tzur et al. (2013), the other five schemes portray how the child operates multiplicatively on composite units, including place value, base ten (PVB10) concepts.

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<tr>
<td><strong>Multiplicative Double-Counting (mDC)</strong></td>
<td>To find a total of 1s in a collection of equal-size each?</td>
<td>T: How many cubes are in 4 towers, 3 cubes each?</td>
</tr>
<tr>
<td></td>
<td>composite units, count C1: (Utters accruing totals while raising simultaneous accrual of fingers for units of 3) 3; 6; 9; 12.</td>
<td>C1: 1-tower-is-3-cubes, 2-is-6, 3-is-9, 4-is-12.</td>
</tr>
<tr>
<td></td>
<td>composite units and of 1s; C2: 1-tower-is-3-cubes, 2-is-6, 3-is-9, 4-is-stop when all composite units 12 were counted.</td>
<td>C3: 2-towers are 6; then double the 6 for 2 more towers. 12 cubes in all.</td>
</tr>
<tr>
<td><strong>Same-Unit Coordination (SUC)</strong></td>
<td>Focus on composite units in T: You have 9 towers. I have 4 towers (all two collections, not losing with 3 cubes each) How many more towers sight of 1s in them, find sum or difference of composite units.</td>
<td>C2: (Count-up-to with fingers) 4; 5-6-7-8-9.</td>
</tr>
</tbody>
</table>

(Count fingers) I have 5 more towers.
We have 39 towers. I did, 9*3=27; 4*3=12; 27+12=39.

To find the difference in 1s between two collections of composite units, identify numerical similarities/differences and anticipates two different solutions.

C1: (Difference-first) I did 9-4=5; 5*3=15.
C2: (Total-first) I did 9*3=27; 4*3=12; then 27-12=15.
C3: (Erroneous) 9-4=5. So, 5 more cubes.

Find total of composite units when given a collection of more cubes, which you also put into towers composite units and more 1s, of 3. How many towers in all?

C1: (Draws groups of 3 dots until having 12; counts groups) 9; 10-11-12-13 towers.
C2: 12 cubes make 4 towers of 3; 9+4=13.
C3: (Erroneous) I add 12+9, so 21 towers.

Given a composite unit (total of 1s), and a smaller composite unit, reverse the double-count process to find how many times the smaller unit fits in the total.

C1: I can make 4 towers, because (raises a finger for 3s, then counts fingers) 3-6-9-12. C2: One-tower-is 3, 2-is-6, 3-is-9, 4-is-12; 4 towers.
C3: (Erroneous) I did 12-3=9.

Given a composite unit (total of 1s), and the number of many cubes would be in each tower? equal-size units into which to distribute it, reverse the one cube to each tower, that’s 4 cubes; double-count process to find another cube to each is 8; then 3-3-12.

The number of items in each. C2: (Erroneous) I did 12-4=8.

Table 3: Six schemes for multiplicative reasoning

I stress two points about schemes for multiplicative reasoning. First, in spite of similar behaviors, mDC is not repeated addition. As a form of additive reasoning, repeated addition involves no change of unit. In contrast, distributing four 1s into each of 3 units while counting both involves such a change, with one unit serving the intermediate, measuring role (e.g., 3 towers * 4 cubes-per-tower = 12 cubes). Repeated addition may help a child obtain an answer. However, the crucial “look-for” in mDC is the child’s intentional coordination of simultaneous operations on 1s and composite units that yield a unit change. Second, one may argue that SUC involves additive operations on composite units. We include it within multiplicative reasoning because the child needs to focus on composite units while being cognizant of the 1s that constitute each. The erroneous answer of C3 illustrates the challenge such a dual-focus presents for children.

A Progression in Fractional Reasoning

A central premise of the progression of schemes for fractional reasoning is that a fraction is not part-of-whole. Rather, a child needs to construct unit (1/n) and
non-unit (m/n) fractions as multiplicative relations (measures; see Simon, Placa, Avitzur and Kara, 2018). This progression builds on operations afforded by a child’s strong concept of number as composite unit, including iteration of a unit to produce and measure a (partitioned) whole as a composite of equal-size units. Figure 1 shows a 3-part whole (A), with the middle part further partitioned into two equal parts. Under Unit A it shows Unit B, which is equal in size to the one above it in A.

Figure: A unit (B) measuring another unit (Whole A = 1) in a 1-to-6 relation

We note that Unit B is not a part of A. It also has not been produced by dividing A, or any other whole. In spite of not being part of any whole, and the 4 visible parts on Whole A being unequal, one can conceptualize B as 1/6 of A. This is rooted in anticipating that B is iterated 6 times to measure A: B fits twice within a sub-unit (‘third’), and we can imagine iterating a pair of Bs, three times, to determine a 1-to-6 relation between B and A. In this process of reasoning one applies her concept of 6 as a composite unit, constituted of six 1s as well as 3 units of 2 units of 1, to create a partitioning operation within a given whole. In Table 4, I refer to Figure 1 when illustrating eight schemes for fractional reasoning – the first four based on iteration and the last four based on recursive partitioning (e.g., 1/2 of 1/3).

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Units/Operations</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Equi-</strong></td>
<td>Anticipate: (a) iterating a unit n times to fit within a whole constitutes a 1-to-n relation; define 1/n as a measure; (b) inverse relations among unit fractions (1/n &lt; 1/m if n &gt; m, as it fits within the whole more times).</td>
<td>T: What fraction is Stick B of Stick A? C1: (Copies Unit B and iterates it 6 times) It’s 1/6; Whole A is 6 times as much as B. C2: (Erroneous) 1/4; there are 4 parts in A. T: If I gave you Unit (C) and told you it is 1/7 of Whole A – how could you check it? C1: I would copy Unit C and figure out if it fits in (measures) Whole A exactly 7 times. T: Will C be smaller or larger than B? C1: To fit C 7 times in Whole A it must be smaller than B, which fits only 6 times. C2: (Erroneous) I say 1/7&gt;1/6, because 7&gt;6.</td>
</tr>
<tr>
<td><strong>Partitioning Scheme (EPS)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Unit fraction scheme and inverse relation</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Partitive Fraction Scheme (PFS)</strong></td>
<td>Anticipate: Iterating a unit fraction (1/n) m times creates a non-unit (composite) fraction (1/n * m = m/n); affords adding/subtracting unit and non-unit fractions &amp; multiply non-unit</td>
<td>T: Sam and Pat each ate 1/6 of a pizza. What fraction of a pizza did they eat in all? C1: 2/6; it’s twice as much as a unit that fits 6 times in the whole pizza. T: What if Ned ate 3/6 of the pizza? C1: Altogether it’s 5/6 of a pizza; adding 2/6+3/6 does not change the kind of unit fraction; Ned ate 1/6 more than they did.</td>
</tr>
</tbody>
</table>
fractions by a scalar (within the whole).

T: What if 3 friends ate 2/7 of a pizza each?
C1: They eat 3 times as much as 2/7, so 6/7.

T: What if 4 friends ate 2/7 of a pizza each?
C1: It’s 4 times as much as 2/7, so 8/7 in all.
C2: It’s also 1 whole pizza and one more 1/7 (writes 1 1/7); they are the same.
C3: (Erroneous) Not sure; maybe 7/8, because we have 8 and 7, but the numerator cannot be larger than the denominator.

Reversible Fraction Scheme (RFS)

Reverse operations used to produce a non-unit fraction (m/n) to recreate the original whole; first “undo” m iterations to produce 1/n; then undo partitioning by iterating it n times (n/n).

T: My candy bar (draws a linear bar) is 7/10 of yours; can you draw your candy bar?
C1: (Partitions the 7/10 into 7 equal units) This little piece is 1/10 of the candy bar; I repeat it 10 times to make the whole.
C2: (Erroneous) 7/10 means 10 parts, so I divide your bar by 10; … I don’t know.

T: I bought a shirt for $24. It’s 4/5 of the original cost. What was the original cost?
C1: Divide 24 by 4 to find $6 as 1/5 of the original price; then multiply $6*5. So, $30.

Recursive Partitioning Scheme (RPS; Unit-of-Unit)
The child anticipates the result of partitioning a unit fraction (e.g., 1/n of 1/m) as if it is applied to every part of the first partition to determine the relation of the resulting unit fraction to the whole.

T: Think of stick A (Figure 1) as a 3-slice pizza. You eat half of the middle slice. What fraction of a whole pizza did you eat?
C1: (Draws marks for halving all thirds) I’ll have 6 slices fit within the whole, so one of them is 1/6 of the pizza.
C2: Two halves in each 1/3rd means 2*3=6; six slices fit in the whole pizza (it’s 6 times as much as my slice).
C3: (Erroneous) 4 parts so mine is 1/4 of the whole.

Unit Fraction Composition (UFCS)

Coordinate RPS-FPS; anticipate result of taking a non-unit fraction of a unit fraction (k/m of 1/n); figure out resulting unit-of-unit, then multiply by the numerator.

T: I ate 4/5 of one slice (1/3rd of Whole A). What fraction of the whole pizza did I eat?
C1: (Marks five 5ths in each of the thirds and colors four in each) A tiny piece is 1/15 of the whole, and you ate 4 of those, so 4/15.
C2: If you ate 1/5 of 1/3 it would be 1/15; but you ate 4, so it’s 4*1/15=4/15.

Distributive Partitioning (DPS)
Recompose a non-unit fraction from unit fractions created by sharing a given number of wholes.

T: Share 3 pizzas equally among 4 of us. What fraction of one pizza each of us gets?
C1: (Draws 3 circles, marks 4 quarters in each) I colored my portion as 1/4 of each pizza. So, 3 times 1/4 in all, or 3/4.
Extend RPS to composite fractions \((k/m)\) by applying it to a unit fraction of it \((1/n \text{ of } k/m)\) and later to any fraction of a fraction \((b/n \text{ of } k/m)\).

T: We have two unpartitioned thirds of the pizza (Whole A in Figure 1). We save 1/4 of those 2/3 for a friend. What fraction of the pizza will our friend get?

C1: I know 1/4 of 1/3 is 1/12 of the whole. The friend will get twice as much, so 2/12.

T: What if 3/4 of the 2/3 of our pizza?

C2: It’s 3 times that 2/12, so 6/12. Also 1/2!

**Table 4: Eight schemes for fractional reasoning**

I point out three key aspects of the 8-scheme progression that, along with teachers, we learned are in need of special attention. First, all schemes beyond the first (EP) depend on the child’s conceptualization of unit fractions as a multiplicative relation (measure) and on the mDC scheme (Table 3). Second, to an adult who knows both PFS (proper) and IFS (improper), advancing to IFS may seem a clear-cut extension of PFS. As teachers reported to us, this is not the case. Conceptualizing the iteration of a unit fraction beyond the whole requires anticipating that a unit fraction, once conceived of through iteration within a given whole – can be “freed” from that whole (Hackenberg, 2007; Tzur, 1999). That is, in IFS the child has to distinguish two kinds of iterations: One used to produce a unit fraction, and another to produce any-size composite fraction. Third, promoting recursive partitioning schemes support students’ understanding of fraction equivalence and decimals/percentages. As the examples (Figure 1) illustrate, 2/6 is equivalent to 1/3 because they are the same quantity (relation to the whole) in the sense of the whole being 3 times as much as each. If we constrain partitioning to numbers that are powers of 10, we create 1/100 (hence, 0.01 and 1%) as 1/10 of 1/10 of the whole.

**Reality Check: Findings of Educational Impact**

I present findings from a few, recent studies published at conferences. The part of my work with colleagues and teachers that focused on growth in students whose educational system designated as having “learning disability in mathematics” is found elsewhere (Hunt, Tzur and Westenskow, 2016; Xin et al., 2017).

**Multiplicative reasoning findings**

Table 5 shows data (Tzur et al., 2017) about the conceptual links articulated between the strength of a child’s concept of number and her capacity to reason with the mDC scheme. The key difference is between Counting-on (weak concept of number) and BAMT (strong), with Doubling explained to have an intermediate strength. ANOVA shows these differences are significant \((F_{2,31}=9.64, p=.001)\), with Bonferroni post-hoc indicating statistically significant differences between counting-on and BAMT \((p=0.001)\), but not between those two and doubling.

<table>
<thead>
<tr>
<th>mDC Correct</th>
<th>Counting-On (N=14)</th>
<th>Doubling (N=8)</th>
<th>BAMT (N=10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%</td>
<td></td>
<td>44%</td>
<td>63%</td>
</tr>
</tbody>
</table>

Table 5: Correlating a child’s concept of number with mDC
Table 6 shows data (Tzur, Johnson, et al., 2018) about the impact of our PD effort on 3rd graders’ reasoning with the mDC scheme, comparing correct solutions of participating (treatment) and non-participating (control) teachers within one school. These results reflect our PD emphasis on teachers’ promotion of the concept of number and of mDC in practice. The two-way ANOVA, comparing year-end to year-end ($F_{1,142}=4.42, p=.037$) and year-start to year-end ($F_{1,203}=6.26, p=.013$), shows interaction in favor of the treatment students in each comparison.

<table>
<thead>
<tr>
<th>Year-End 2016</th>
<th>Year-Start 2016/17</th>
<th>Year-End 2017</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>25%</td>
<td>8%</td>
</tr>
<tr>
<td>Control</td>
<td>36%</td>
<td>19%</td>
</tr>
</tbody>
</table>

Table 6: Impact of PD effort on 3rd graders’ reasoning with mDC scheme

Table 7 shows data (Tzur, Wei, et al., 2018) about the conceptual links articulated for the SUC scheme as a conceptual precedence to MUC – determined by her responses to six items, including three that focus on place-value, base-ten (PVB10). ANOVA shows statistically significant differences between students who did not have the SUC scheme and those who did. Correlations (Pearson) between two sub-scales of this instrument, one focusing on MUC and another on PVB10, are shown.

<table>
<thead>
<tr>
<th>No SUC</th>
<th>Yes SUC</th>
<th>MUC-Direct vs. PVB10</th>
</tr>
</thead>
<tbody>
<tr>
<td>4th Grade (N=200)</td>
<td>5th Grade (N=351)</td>
<td>All (N=551)</td>
</tr>
<tr>
<td>9%</td>
<td>19%</td>
<td>14%</td>
</tr>
<tr>
<td>29%</td>
<td>41%</td>
<td>37%</td>
</tr>
<tr>
<td>$F_{1,199}=35.1, p&lt;.0005$</td>
<td>$F_{1,350}=54.0, p&lt;.0005$</td>
<td>$F_{1,550}=94.7, p&lt;.0005$</td>
</tr>
<tr>
<td>$r = .47, p&lt;.0005$</td>
<td>$r = .46, p&lt;.0005$</td>
<td>$r = .48, p&lt;.0005$</td>
</tr>
</tbody>
</table>

Table 7: Correlating a child’s SUC scheme with her MUC scheme

**Fractional reasoning findings**

Table 8 shows data from preliminary analysis comparing 3rd and 4th graders’ outcomes on a validated, 6-item sub-scale (incorrect=0, correct=1) for measuring students’ reasoning with the Equi-Partitioning (EP) scheme (unit-fractions, 1/n). The treatment group included one grade-3 class (N=28) taught by their homeroom teacher (with my guidance). For control we used a grade-4 class (N=25) at the same school of the treatment class, and a grade-3 class (N=20) at a different school with similar student demographics and the same principal. Independent sample t-tests show: (a) grade-4 (control) outperforming grade-3 (control) as expected ($t=3.8, df=43, p=.001$); (b) grade-3 (treatment) outperforming grade-3 (control; $t=2.9, df=46, p=.006$); and (c) no statistically significant differences between grade-4 (control) and grade-3 (treatment; $t=.94, df=51, p=.35$). Combined, these results suggest that instruction focusing on constructing unit-fractions as
a multiplicative relationship (measure) promote students’ reasoning and outcomes more than instruction focusing on unit fractions as “parts-of-wholes.”

<table>
<thead>
<tr>
<th>Treatment (3rd, N=28)</th>
<th>Control (3rd, N=20)</th>
<th>Control (4th, N=25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct (avg.)</td>
<td>71%</td>
<td>52%</td>
</tr>
</tbody>
</table>

Table 8: Impact of teaching unit-fractions as measure vs. part-of-whole

Concluding Remarks

I have presented a cognitive progression of schemes distinguished through research with children and provided some evidence of the applicability of this progression to practice. In this section, I note three key implications to our understandings of the intricate relationships between learning and teaching of elementary mathematics.

First, articulating how schemes in the progression provide a cognitive foundation for mathematical contents and teaching methods – points out a viable alternative to typical approaches for designing and implementing curricula. The starting point and logic underlying typical approaches seem to be adults’ mature knowledge. Formulation and sequencing of mathematical goals for students’ learning (e.g., “standards”) reproduce this knowledge, and teaching methods appear as ongoing attempts to fit students to the curriculum. Instead, the alternative approach espoused by the progression I presented challenges teachers to continually fit the curriculum – what and how they teach next – to every child’s evolving cognition. Following Steffe’s (1992) notion of adaptive teaching, I termed this alternative approach Student-Adaptive Pedagogy. I also refer to it as “Steffe in the Classroom.” An instructive example of this approach would be teaching PVB10 concepts to students who have constructed not only a strong concept of number but also the first four schemes for multiplicative reasoning (see Tables 3 and 8). Our data indicate this may not be the case for a large portion of 4th and 5th graders, whereas typical approaches attempt to fit first graders to learning PVB10.

Second, often distinguished through research in lab settings, the schemes in this progression show great promise for informing and improving practice. In the past few years, my PD efforts focused on promoting and studying teachers’ learning of this progression and using it to support their shift to student-adaptive pedagogy. These efforts convinced me that (a) teachers are both interested in and capable of such learning and (b) the process of teaching the teachers is lengthy, challenging – and highly rewarding to them and to their students (as seen in the results above).

Third, my research on adults’ knowledge (Tzur, 2019) and learning of unit fractions indicate that the progression articulated with children, as well as methods to promote particular schemes, seem applicable also for adults (e.g., teachers). My work as a teacher educator, though not systematically studied, indicated this is also true for other schemes. For example, constructing the mDC
scheme in place of “repeated addition,” or MUC to anchor understanding of PVB10, or recursive partitioning schemes to make sense of fraction concepts (e.g., equivalence, additive operations on unlike denominator fractions, the meaning of decimals and percentages) – all proved direly needed for the teachers themselves. Teachers’ construction of those schemes then promoted their understanding of the progression while clearly distinguishing one’s own mathematics from that of her students.

References


**IMPROVING EARLY NUMBER LEARNING IN CONTEXTS OF DISADVANTAGE**

*Hamsa Venkat*

**Abstract**

In this plenary address, I provide an overview of an intervention focused on improving early number learning in a South African context of disadvantage involving large classes and limited resources. This overview takes in the ways in which context and conditions informed the intervention’s focus on tasks, teaching and resources that emphasised attention to number structure. In the paper, I note a trajectory of writing that has attended, in the aftermath of evidence of improvements in student working with number structure, to what mathematical structure entails, and what teaching for connection and mathematical structure consists of. The presentation, therefore, represents an attempt at a more holistic story-telling of design-based research than papers typically allow.

**Keywords:** early number learning, progression, mathematical structure, early number teaching, South Africa

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The length restrictions on papers in mathematics education, across journals and conference papers, typically allow us to tell rather circumscribed stories – the impact on learning, or the nature of teaching, or the conditions of schooling. In practice though, these things occur together, and therefore have to be considered together in the design and evolution of interventions that seek to effect change in learning outcomes. In this paper, I seek to bring these aspects together in the telling of a story of an intervention programme that we implemented in the early grades in South Africa, that sought to improve early number learning outcomes for children in contexts of socio-economic disadvantage.

This story pulls together elements that we have dealt with rather separately across a range of prior publications, where one or other aspect takes up the foreground and the other elements form the backcloth. In this pulling together, my aim is to share details and rationales for the trajectory of the work, and to show how the ‘pieces’ that we have published fit together to provide an exemplar of research-based development in a developing country context. This is important given the point made by Ole Skovsmose (2011): that the research base in mathematics education is heavily biased towards research carried out in contexts of advantage, with limited research findings emanating from countries and contexts of disadvantage. This means that less is known either about the developmental trajectories of children in these contexts, or about how to support their learning and development.

A recent United Nations report (Guterres, 2018) on progress towards the international Sustainable Development Goals suggests that this bias towards advantage further exacerbates the already serious challenges of access to good quality education: of the 617 million global population of primary and lower secondary children, the report indicates that 58% of this group are not achieving at minimum proficiency levels in reading and mathematics.

I begin with an outline of a project – the Wits Maths Connect-Primary project – located in the University of the Witwatersrand in Johannesburg, and the terms and aims of this project, related to the conditions in the broader primary mathematics landscape in which it was located. It was our study of classroom environments in the ten partner schools that led us into choices of interventions; my focus in this paper is on the intervention content and format that we chose for focus on early grades’ number learning, and the evolution of this model over time, informed by studies of aspects of mathematics, of teaching, and of learning outcomes and the broader policy landscape.

**The Wits Maths Connect-Primary project**

The Wits Maths Connect – Primary project, began in 2011 with an initial five year-remit to design, develop and implement research-based interventions in ten
partner state primary schools and study their outcomes. The terms of the project were that it should be located in a university, that it should establish a postgraduate research team, and focus on designing and studying interventions to improve learning outcomes that would be possible to implement at a larger scale in the context and conditions of government schools in South Africa. Following positive evaluations in 2015, the project is currently in its second five-year phase, with six of the ten partner schools from Phase 1 continuing into Phase 2.

This project came about as part of a call, in 2009/2010, for research and development Chairs in mathematics education that could address poor learning outcomes in mathematics at all phases in South Africa. Two proposals for projects in primary mathematics were accepted, one of them, ours. In reading the literature base available on primary schooling in South Africa as we began the project, much of this carried out from ‘generalist’ policy and curriculum perspectives, we were aware of some key currents. Firstly, there was writing on waves of post-apartheid curricular reform:

- an initial move to an ‘outcomes based curriculum’ that provided limited specification of content and progression, that was criticised soon after its implementation for exacerbating, rather than addressing, a context marked by high levels of inequality that continued to be intercut by the racial divisions on which the apartheid system had been premised (van der Berg, 2005).

- two subsequent waves of return to increasingly closely specified curricula, with the version implemented in 2011 prescribing content, sequencing and pacing on a weekly basis, supplemented in some provincial initiatives with individual lesson plans. National workbooks for students were introduced alongside the move to a more prescriptive curriculum sequence, seeking to address concerns related to the limited availability of textual resources, and poor usage of the resources that were available.

- across the critiques, a recurring message was related to poor conceptual understandings of mathematics amongst primary teachers, feeding into widespread reporting of poor progression (Reeves and Muller, 2005) and highly rote approaches to teaching and learning.

There was more limited evidence at a larger scale focused specifically on mathematics, but a key recurring finding, across multiple studies, was related to the prevalence of highly inefficient counting based approaches to number working (Schollar, 2008; Hoadley, 2006). Schollar’s (2008) data gathering in the middle grades provides graphic illustrations and commentary on what this phenomena looked like in students’ work, detailed in Figure 1:
This kind of working was set within what has been described as ‘a bi-modal’ system: a smaller system, serving middle class children, that is predominantly well-resourced and well-functioning, with high expectations for learners, and a much larger system, serving poor, working class urban and rural children. Fleisch (2016), summarising the features of the latter system, describes instruction in these schools in the following terms:

‘low expectations, narrow repertoires of pedagogical techniques, slow-paced lessons, incomplete coverage of the curriculum, and arbitrary sequencing’ (p. 441)

Evidence from national and regional assessments shows that reading, writing and mathematical skills in these contexts are very low (Spaull, 2013), with the majority of learners failing to meet curricular expectations. These patterns of low expectations and low-level tasks have been described as part of the classroom experience for marginalised students in many parts of the world, including the developed world (e.g. Haberman, 2010). But this similarity is underscored by important differences: class sizes in the early grades classes we work in range from 40 to over 70, and classrooms in the urban township schools in our partner school group are frequently temporary ‘container’ structures that are small and overcrowded with three or four learners seated at a two-place desk. Teacher absenteeism is relatively common. In Johannesburg, substantial South African and pan-African inward migration also leads to most schools being multilingual. Suburban schools situated in historically white areas largely work with English as the language of instruction from the start, even though many of these schools now serve substantial numbers of black African learners; township schools, often organise for different classes in a grade to use different South African home languages as their language of instruction. Typically, this means that a township school with 5 classes may have two Sepedi classes, two Xitsonga classes, and one isiZulu class. A further important point relating to sub-Saharan African classroom settings is writing pointing to a ‘tissue rejection’ of various policy initiatives that have tried to promote learner-centred pedagogic approaches in cultures that tend
to view teaching in ‘authoritative’ instructional terms, rather than in more constructivist or negotiated terms (Tabulawa, 2013). Two, somewhat contradictory currents run through the research on instruction in these contexts: firstly, there are reflections on what has been described as an ‘absence’ of pedagogy – instruction involving poor sequencing of content and unresponsive to student offers (Hugo and Wedekind, 2013); on the other hand, there is also ample evidence of the importance of good quality instruction in contexts where there are few, if any, options for ‘second sites’ of access to disciplinary learning.

This combination of evidence on learning outcomes, instruction, classroom context and culture had to be taken into account in our thinking about what to investigate in terms of the teaching and learning of early grades’ mathematics in our partner schools, and how to go about investigating these phenomena. These investigations could then inform our design of research-based interventions to try and improve outcomes.

**Investigating mathematics learning in WMC-P partner schools**

Two features of the broader landscape led to a decision to narrow our focus to early number teaching and learning. Firstly, as pointed out already, an extensive body of evidence pointed to a lack of progression beyond what van den Heuvel-Panhuizen (2008) describes as ‘calculating-by-counting’. Secondly, and in line with the mathematics education research base pointing to early number understanding as fundamental to mathematics learning, the South African early grades’ curriculum allocated more curriculum time to number than to the other topic areas (DBE, 2011). Curricular emphases on number ranges from 65% in Grade 1 to 58% in Grade 3 (age appropriate students in these grades are 7-9 years old, but grade retention based on lack of adequate progress means that many classes include older children as well). This emphasis, reflected in the assessments of progress, meant that making a difference in number learning, would contribute substantively to overall performance.

But knowing what to focus on does not dictate how to go about understanding the phenomena on the ground. Here, the research base was dealt with in combination with pragmatic issues related to our work in the partner schools. On the research side, poor levels of reading and writing meant that a written test was likely to provide limited evidence of number understandings in the early grades. Related to this, the research base also suggested that a key facet of importance in early number working was not simply whether a child could produce a correct answer in an additive situation, but instead, how that answer was produced. Askew and Brown (2003) summarise these options in a progression that incorporates: unable to count all, count all, count on, count on from larger, work with known fact and use number properties to derive further facts. On the pragmatics of access to schools, we had initially hoped to begin our baseline explorations with the 2011 Grade 1 cohort. However, schools indicated to us that the majority of the students
entering the ten schools came with no prior experience of education, leading to a focus on school readiness work at the start of Grade 1 that we did not wish to disrupt. This led us to shift our attention in 2011 to the Grade 2 cohort, and a decision to work with individual interview-based assessments with a cross-attainment sample of six students (two identified, by their class teachers, as low attaining, two mid attaining and two high attaining in each partner school). (As an aside, Grade R classes started to be introduced across all partner schools during the course of our Phase 1 (2011-15) activities.)

We chose to work with the Mathematics Recovery assessments designed by Bob Wright and his colleagues in Australia (Wright, Martland and Stafford, 2006), drawing from the work of Steffe, Cobb and von Glasersfeld (1988) on the trajectory of counting strategies into efficient working with number structure. While these assessments encompass attention to numeral recognition and identification, counting, additive and early multiplicative working, we largely omitted attention to early multiplicative working given the length of time that we saw early Grade 2 children were taking to work with the other aspects – frequently 40-45 minutes. In the five Phase 1 township partner schools, translators accompanied the researcher and translated test items and learner responses. Wright et al. (ibid.) provide details on coding the sophistication of children’s responses across a range of aspects within what they describe as the Learning Framework in Number (LFIN). For the purposes of this paper, I summarize the centrepiece of this framework – the Stages of Early Arithmetical Learning (SEAL) (Table 1), because a key part of the development of our work has been related to thinking about this trajectory of working from a base in counting into number structure.

<table>
<thead>
<tr>
<th>Stage</th>
<th>Title</th>
<th>Description of counting/calculating strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Emergent count</td>
<td>Cannot count a visible collection of counters</td>
</tr>
<tr>
<td>1</td>
<td>Perceptual count</td>
<td>Can count perceived items (seen, heard or felt) and solve additive tasks involving displayed collections</td>
</tr>
<tr>
<td>2</td>
<td>Figurative count</td>
<td>Can solve additive tasks involving one or two screened quantities using the ‘count all’ strategy</td>
</tr>
<tr>
<td>3</td>
<td>Initial number sequence</td>
<td>Can solve addition tasks using ‘count on’, ‘count-up-to’ and ‘count-down-from’</td>
</tr>
<tr>
<td>4</td>
<td>Intermediate number sequence</td>
<td>Can use ‘count-down-to’ to solve missing subtrahend tasks and can choose the more efficient of ‘count-down-from’ and ‘count-down-to’ for task at hand</td>
</tr>
<tr>
<td>5</td>
<td>Facile number sequence</td>
<td>Can use a range of ‘non-count-by-one’ strategies involving calculation-by-structuring (doubles, near</td>
</tr>
</tbody>
</table>
Table 1: A summary of Wright et al’s (2006) ‘Stages of Early Arithmetical Learning’

These stages can be viewed in different ways depending on the theoretical framing that is applied. In van den Heuvel-Panhuizen’s (2008) terms, the break between calculating-by-counting and calculating-by-structuring occurs between Stages 3 and 4. In Sfard’s (2008) distinguishing of processual and objectified approaches based on reification, Stage 3 is important as the first marker of some reifications of counting processes into number objects within count-on strategies. More broadly, there are disputes on whether the trajectory from counting into structuring works at all, with writers working in the approach outlined by Davydov (1992) using tasks geared towards structural quantitative relations from the start, rather than starting with counting (e.g. Schmittau, 2003). Working with Davydov’s approach, counting-based working would likely be viewed as part of an ‘everyday concepts’ repertoire, while structure-based working is viewed as part of the realm of ‘scientific concepts’.

Baseline studies of teaching and learning in WMC-P schools

Working with a set of schools in one Johannesburg district, selected by the district Foundation Phase Mathematics adviser on the basis of ‘underperformance’, we began our work in 2011 with two requests to the partner schools: firstly that we observe a Mathematics lesson with all the Grade 2 teachers across the ten Phase 1 partner schools, and secondly that we conduct the interview-based LFIN assessment with six children in each school sampled as described above. These two requests were made in order to understand our ‘baseline’ and the ways in which it might overlap and differ with the broader South African evidence sketched earlier. Two key insights emerged from our analyses of these two datasets. At the learning level, the prevalence of unit counting was confirmed. In applying Wright et al’s SEAL stages to the learner response data, we found that 75% of the children in the cross-attainment sample (based on the six schools that have participated across both Phases of the project) were using, at best, concrete count-all strategies for additive problems where the whole value was less than 20. A further 8/36 (22.2%) children were able to use ‘count-on’ strategies, with only 1 child in the sample showing any evidence of working with number properties. We found, further, that there were gaps in children’s working with counting sequences, with nearly a quarter of the learners in the 2011 sample unable to state the number word after a given number in the 1-10 range without dropping back to 1, with this proportion rising to over a third unable to state the number word before a given number in the same range. In Davydov’s terms, this evidence suggested poorly developed everyday concept understandings of numbers alongside bigger gaps in scientific concept understandings of number relationships.
On the teaching side, our findings added mathematical nuance to prior findings of gaps in teacher knowledge and poor understandings of progression. Specifically, analyses of teaching pointed, more seriously perhaps, to evidence of breakdowns in coherence in the ways mathematics was mediated in the early grades. By way of example, Venkat and Askew (2012) described instances of ‘unstructured’ working with ‘structured’ artifacts – teachers working with abaci for single digit addition, for example, but making no reference to the base 10 structural affordances of the abacus in their working. Structure was also sidelined in the repeated return to unit counting, negating possibilities of working with the relationships between established results and further results derive-able from these. This localization had been noted previously in the ongoing provision of counting resources and in instruction (Ensor et al., 2009).

Interventions and outcomes

On the basis of this evidence, we developed number ‘ lesson starter’ activities and supporting resources and professional development (PD) focused, initially, on developing more fluent and flexible counting alongside attention to the number relationships and properties that are required for progression to more efficient calculation. On the pedagogical side, this involved encouraging teachers to build coherent and connected repertoires in their mediation of mathematics via tasks and examples and between artifacts, inscriptions and teacher talk.

The PD model across 2011-2018 has involved a termly teacher workshop focused on the progressions outlined in Wright et al’s SEAL model with tasks, activities and resources shared and discussed in the workshop, followed by in-class coaching/observation of teachers’ work with these activities. Having begun in 2011 with a Grade 2 cohort, we followed this cohort through into Grade 3 in 2012, moving our associated PD work to the 2012 Grade 3 teacher group. We then picked up a new Grade 1 cohort in 2013, and followed them through to Grade 3 similarly. The same set of LFIN assessments used in 2011 were administered to the 2014 Grade 2 cohort.

As noted already, the extension into a second phase allowed for a new Grade 1 cohort to be picked up in 2017 and followed into Grade 2 in 2018, with the same assessments administered again early in the grade. With six schools from Phase 1 continuing into Phase 2, by 2018, we had a quasi-longitudinal picture of shifts in attainment across these schools. The detail of the SNS project and the shifts in patterns of performance has been detailed in other writing (Venkat et al., 2019); here, I provide a summary of the outcomes recorded (see Table 2) and a brief commentary on the changes seen over time, and go onto focus on the additional research questions that were stimulated along the way by this evidence of impact.
<table>
<thead>
<tr>
<th>SEAL Stage</th>
<th>2011</th>
<th>2014</th>
<th>2018</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4 (11.1%)</td>
<td>1 (2.8%)</td>
<td>1 (2.8%)</td>
</tr>
<tr>
<td>1</td>
<td>8 (22.2%)</td>
<td>4 (11.1%)</td>
<td>1 (2.8%)</td>
</tr>
<tr>
<td>2</td>
<td>15 (41.7%)</td>
<td>9 (25%)</td>
<td>10 (27.8%)</td>
</tr>
<tr>
<td>3</td>
<td>8 (22.2%)</td>
<td>20 (55.6%)</td>
<td>15 (41.7%)</td>
</tr>
<tr>
<td>4</td>
<td>1 (2.8%)</td>
<td>2 (5.6%)</td>
<td>7 (19.4%)</td>
</tr>
<tr>
<td>5</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>2 (5.6%)</td>
</tr>
</tbody>
</table>

Table 2: SEAL Stage distributions, 2011-2014-2018, based on n=36 students in each of these years

The key shift seen in 2014 was focused on the Stage 2 (count-all) - Stage 3 (count-on) transition: whereas 75% of the sample were functioning at the count-all at best stage in 2011, by 2014, this proportion had dropped to 38.9% of the sample. Predictably, this shift from count-all to count-on based working was accompanied by improvements in awareness of forward and backward number word sequences, with the proportions able to deal with stating the number word after a given number in the 1-10 range rising from just over three quarters of the sample in 2011 to over 95% in 2014.

These improvements in counting facility led us, in the second phase, into what we have described as a ‘straight for structure’ PD focus informed by Davydov’s (1990) approaches. Essentially, we directed attention, from Grade R and Grade 1, directly into focus on number relationships in our lesson starter activities, knowing that the curriculum would continue to provide attention to counting alongside this. This adapted design appears to have contributed to a further shift in the 2018 patterns of performance focused on the Stage 3-Stage 4 transition: while in 2014, only 5.6% of the student sample showed evidence of facility with working with number relationships and properties, in 2018, this proportion had increased to 25% of the cohort.

**Further research stimulated by improved learning outcomes**

The improvements seen in learning outcomes in 2014 led us back, in a context where formal schooling has been described as the single site of access to disciplinary learning for children from disadvantaged background, into looking at shifts in early number teaching that might have contributed to the changed profile of performance. We had the baseline video data from one Grade 2 lesson for each teacher described earlier, and similar baseline video from Grade 3 teachers in 2013. Following this we had, where teachers once again gave informed consent, video data of lesson starters teaching across Grades 1-3 in 2013-2015. This video dataset included more extended instances of earlier and later teaching by some teachers in the partner schools, so we began an analysis focused on twelve earlier/later lessons involving six teachers. Our initial sense was that the later teaching was generally more coherent and connected than the earlier teaching, though sometimes this coherence and connection was only manifest in some
teaching episodes. But while the notion of ‘connection’ has been widely discussed in the mathematics education literature, coherence, in the sense of logical bringing together of artifacts, inscriptions and talk, was much less in focus – perhaps because this is much less an issue in the advantaged contexts that predominate in the literature base. Connections, for us, were viewed as a key route to enlarging the example spaces in which students were able to operate, and provided a key mechanism for access to mathematical structure and increasing generality. On the instructional side, the gaps in attention to coherence and structure/generality as markers of instructional change towards improved quality led to the development of the ‘Mediating Primary Mathematics’ (MPM) framework for considering the quality of instruction in contexts where these elements could not be taken for granted (Venkat and Askew, 2018). In the MPM framework, the broad trajectory of development involves moves to coherence, and then connection, and then expansion into structure/generality. Askew’s (2019) and Askew et al.’s (2019) writing provides detail on the nature of shifts in teaching in a trajectory among teachers who had been part of the professional development activities described above.

A further move was then to clarify our usage of the term ‘structure’. Kieren (2018) has noted that the lack of clarity on the meaning of this word suggests that ‘structure’ is viewed as ‘if it were tantamount to an undefined term’ in mathematics, at the same time as being widely agreed as important. Clarifying what we meant when we pointed to learning, and to teaching, moving in the direct of increased attention to structure, therefore became a further point of attention in our work (Venkat et al., 2019). Centrally, structure for us, relates to an emphasis on relationships between mathematical entities, numerical relationships in the context of this project, rather than on calculating, which can involve working with numbers more sequentially rather than relationally. Further writing on structural working is currently in process.

Concluding comments

Taken together, this design-research based model has produced a number of useful developmental and research insights. In relation to development, a practical model for working in the conditions of South African schooling for improvements in early number teaching has been studied and reported. Student support materials and teaching resources that have contributed to the gains in outcomes have also been developed and are available in open source format from our project website (https://www.wits.ac.za/wits-maths-connect/wits-maths-connect-primary/structuring-number-starters/). On the research side, what it means to work towards mathematical structure has been clarified, and a framework that is attuned towards coding for changes in teaching in the direction of increased attention in contexts of disadvantage and in a culture of teacher-led working has been developed and used to productive effect.
Our privilege has been to have a ten-year window to develop these insights over time. This timeline that has allowed us to see into various initiatives and their impacts over a longer timeframe than shorter research typically allows. The outcome is a robust set of outcomes and insights, that takes context and culture into account, and attends to learning as part of a broader eco-system that includes teachers, schools and the broader landscape. As such, the work has fed into provincial and national partnerships in ways that research in mathematics education is often criticised for failing to do. Along the way, we have learned more about the possibilities and the challenges of working for improvement in school and classroom environments where many of the ‘taken for granted’ assumptions about conditions and availability of resources that permeate much of the mathematics education literature base are simply not in place.

Acknowledgement: The Wits Maths Connect-Primary project is generously funded by the FirstRand Foundation, Anglo American, Rand Merchant Bank, the Department of Science and Technology and is administered by the National Research Foundation. My thanks also to the team of staff and postgraduate students in the Wits Maths Connect-Primary project who have been involved in the research and development work reported in this paper.

References


EXPLORING THE SEMIOTIC POTENTIAL OF TOUCHTIMES
WITH PRIMARY TEACHERS

_Sandy Bakos and Nathalie Sinclair_

Abstract

In this paper, we investigate teachers’ interactions with an iPad touchscreen application (TouchTimes), which provides a visual and kinesthetic way for children to learn about multiplicative relationships. TouchTimes (TT) offers a functional, and symbolic model of multiplication that does not rely on repeated addition. Using the Theory of Semiotic Mediation, we analyse the semiotic potential of TT for supporting primary students’ learning of multiplication. We also study the semiotic potential of TT for primary school teachers’ learning-to-teach multiplication by analysing their own interactions with this technology, as well as their responses to videos of students using it.

**Keywords**: mobile technology, primary teachers, multiplication, semiotic potential

Introduction

Digital technology is providing new resources and means that show promise in supporting the mathematical learning of young children (e.g. Sinclair and Baccaglini-Frank, 2017). In schools, however, there is often a mismatch between the developers’ intended purpose of technology devices and how they are implemented and used by teachers (e.g. Clark-Wilson, Robutti and Sinclair, 2016). Contributing to this mismatch are obstacles and constraints such as the

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availability (or lack of) resources (both technology resources for student use and professional development or pedagogical resources for teacher use) and the impact of teacher confidence and attitude towards technology and its use in the classroom (Thomas and Palmer, 2016). Many teachers find it very difficult to exploit skilfully the opportunities that technology can offer for learning.

The aim of this research is to examine the semiotic potential of a particular digital technology, TouchTimes (hereafter TT) (Jackiw and Sinclair, 2018), in changing how primary teachers think about multiplication both themselves and in relation to their teaching of this concept to children. We see this as particularly relevant, given the prevalent emphasis on introducing multiplication as repeated addition, a practice encouraged by curriculum documents and embraced in teaching practice (Askew, 2018). Repeated addition is commonly used for introducing and working with multiplication in primary grades and tends to become firmly entrenched as the dominant perception governing multiplicative situations, both for students and for primary teachers. Drawing on ideas both from Vergnaud’s (1983) functional approach to multiplicative thinking and from Davydov’s (1992) change-of-unit approach, the TT design provides users with a tangible, visual and symbolic interaction that encourages different ways of thinking about multiplication.

According to Davis and Renert (2014), although it appears to be very important to provide young learners with varied approaches in early mathematics learning, teachers often select approaches (such as multiplication-as-repeated-addition) that fit particular aspects of a concept. These dominant approaches may be so well rehearsed that they preclude teachers from considering other approaches that could also benefit student learning. In our work with teachers, TT has been presented as a resource that can be used alongside multiplication-as-repeated-addition when introducing the concept of multiplication, in order to offer a directly multiplicative, as well as an additive, perspective on multiplication.

**Theoretical framework**

The theoretical orientation of this study is based upon the Theory of Semiotic Mediation (hereafter, TSM) (Bartolini Bussi and Mariotti, 2008), which combines an educational perspective to Vygotsky’s focus on the significance of signs. The TSM provides a framework for describing and explaining the process of teaching–learning mathematics which begins with the use of an artefact and eventually leads to student acquisition of mathematical content. Although more commonly used in classroom situations, with a focus on student learning mathematical concepts, the TSM is also well suited for our purposes because the focus is on teacher learning to teach mathematical concepts.

Following the distinction made in Rabardel’s (1995) instrumental approach, TSM takes an artefact to be a material or symbolic object, which has been designed to answer a specific need, and an instrument is a combination of the artefact itself
and a cognitive component (Trouche, 2004). Taking a semiotic perspective towards the use of an artefact while completing a task involves “recognizing the central role of signs in the construction of knowledge” (Vygotsky, 1934/1978, p. 60). The production, interpretation and elaboration of signs (which may include spoken and written language, gestures, numbers and diagrams) that emerge while using an artefact to complete a task are important in fostering the generation of higher mental functioning as students move towards ‘knowing how’. This process of “internal reconstruction of an external operation” (p. 56) is referred to as internalization.

Artefact signs, mathematical signs and pivot signs are the three main categories of signs identified by Bartolini Bussi and Mariotti. They are helpful in drawing attention (both for teachers and for researchers) to the co-ordination between bodily actions, tasks and a mathematical discourse. Artefact signs emerge from students’ personal experiences of using an artefact to accomplish a task. Mathematical signs are culturally determined signs related to the mathematical meanings connected with the artefact. The primary goal of the semiotic mediation process involves student production of mathematical signs, with the teacher carefully orchestrating the transition (marked by pivot signs) “from signs that express the relationship between artefacts and tasks, to signs that express the relationship between artefact and mathematics knowledge” (Mariotti, 2012, p. 29). Therefore, in this sense, mediation relates to the relationship that an artefact has with the personal meanings evoked by its use to accomplish a task, as well as the mathematical meanings that emerge from its use that are recognizable as mathematics by an expert. This double semiotic link is significant within the TSM – the semiotic potential of an artefact.

The didactic cycle is an essential component of the TSM in examining teaching and learning in a classroom context from a semiotic perspective. However, for our purposes, when working with primary teachers in a research context, we are specifically interested in the unfolding of the semiotic potential (Mariotti, 2012) of TT, both for learning multiplication and for learning-to-teach multiplication. For this reason, our focus will solely be on the first phase of the semiotic mediation process, which involves the emergence of teachers’ (rather than students’) personal signs in relation to the use of the artefact, in this case TT.

We acknowledge the complexity in using the TSM in a research context with a digital tool which provokes subtle shifts between teacher-as-learner herself and teacher-as-expert (with prior knowledge of mathematics and as an educator of children). These different perspectives were brought to the forefront and must be considered in relation to the teachers’ prior teaching practices of multiplication and their use of TT as an artefact. The teacher-as-learner perspective involves the teacher becoming a participant in the situated and embodied knowledge of multiplication (de Freitas and Sinclair, 2014) through the use TT. The second perspective involves the interaction between the tool’s use and the mathematics
that emerges as *evoked knowledge* (Hoyles, 1993), essential when considering the teacher-as-expert, as someone who already knows the mathematical idea evoked by the artefact but who seems automatically and unconsciously to turn their thinking to the pedagogical possibilities related to student mathematical learning through the use of the artefact.

**Methods**

The data for this paper comes from the first phase of a larger project involving a collaborative process between four mathematics education researchers and six teachers from four primary schools (with students aged 5–13) in Metro Vancouver, Canada. The teachers met with the researchers four times during the 2018–2019 school year, though not all participating teachers were able to attend all of the meetings. Throughout this phase of the project, teachers were introduced to TT and engaged with some tasks proposed by the research team, including ample discussion about TT itself, as well as about the tasks. These discussions included critiquing, suggesting improvements, adaptations or refinements to TT itself and the tasks, making explicit links to relevant curriculum and sharing other potential resources for use in teaching multiplication.

Data for this paper was generated through a video-recording of the third teacher meeting, which lasted 52 minutes and was transcribed for analysis. This meeting involved two of the six teachers, Susan (kindergarten teacher) and Leah (grade 3 teacher), both of whom had experimented with TT in their classrooms and were reporting on their experiences. Although the intended meeting agenda had been to discuss the eight tasks created by the research team, this changed after Susan shared a video of a small group of her kindergarten students interacting with TT. Most of the meeting time was subsequently devoted to discussion prompted by the video.

Before presenting the data that emerged from this discussion, we will first describe TT in general terms and then undertake an *a priori* analysis of the semiotic potential of TT for learning multiplication. In the Results section that follows, we will analyse the semiotic potential of TT for learning-to-teach multiplication, based on the third research meeting described above.

**Brief description of TouchTimes**

In order to visualise TT, it may be helpful to view this short video demonstration (m.youtube.com/watch?v=L3BRXZfBbZo), prior to reading the description that follows. With this application a user can place and hold his or her fingers on one side of the iPad screen to create coloured discs (termed “pips”). (In what follows, we presume that the user has opted for the left side of the screen, but the description is symmetric for the right side of the screen as well.) Each single finger that maintains continuous contact with the left side (LS) produces a different coloured pip (Figure 1a). When a user taps her or his finger(s) on the right side (RS) of the screen, a unit containing coloured discs (termed “pods”) appear
beneath each finger which contains the same-coloured pips being created by the user’s fingers in contact on the LS (Figure 1b). The number sentence that corresponds with the pips and pods being created by the user is also displayed by TouchTimes. The pips are maintained within the pods as long as the user maintains contact with at least one finger on the LS, but if all fingers are removed, the pods disappear (“multiplying by 0”). Screen contact can be made one finger at a time or by multiple fingers simultaneously.

Downton and Sullivan (2017) argue that, “the co-ordination of composite units is the core of multiplication, and that young children’s (8-year-olds) concept of multiplication is based on the meaning they give to the composite units they construct” (p. 306). To think simultaneously about units of one and units of more than one requires an ability to think multiplicatively. TT allows children to create, through fingers and gestures, a multiplicative model that involves the co-ordination of two quantities, similar to Figure 1c below (Boulet, 1998, p. 13). One way to conceptualise this action is to see the pips (via LS contact) which are then unitised into pods (via RS touches), with the pods then unitised into the product, as in the Davydovian approach (see Boulet). Seen this way, the sentence $3 \times 4 = 12$ is read as the multiplicand times the multiplier equals the product $(3)(4\times)$, which reverses the typical North American approach ($3$ groups of $4$). Of course, it is also possible to see the pods as being groups of pips, which can be understood in terms of repeated addition. However, the simultaneous aspect of the two-handed touches retains less of the temporally sequential sense of repeated addition.

Figure 1: (a) Creating pips; (b) Creating pods; (c) Multiplicative model

The semiotic potential of TouchTimes: An a priori analysis

Within the framework of the TSM, we will now conduct an a priori analysis of the semiotic potential offered by TT by co-ordinating the key affordances of TT, which are listed in Table 1, with significant mathematical meanings of multiplication.

<table>
<thead>
<tr>
<th>Semiotic potential</th>
<th>Mathematical meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>The multiplicand must be created before the multiplier.</td>
<td>Focuses attention on unitising as first action.</td>
</tr>
</tbody>
</table>
Each pod is bounded and its shape reflects the configuration of the pips. The pod is a unit and is determined by the particular choice of pip touches.

Changing the number of pips affects the composition of all of the pods. A new pip “spreads” across each pod—showing multiplication as functional.

Each pip is coloured differently. Draws attention to the multiplier effect.

Pods cannot be made until there are pips. The second unitising can only occur after the first unitising.

The pip fingers cannot be removed from the screen. Zero times anything produces nothing.

Numerals appear on the screen that reflect the number of pips and pods. Symbolic representations both of the multiplicand and of the multiplier.

An equation appears after both pips and pods have been created. A multiplicand and a multiplier together produce a product.

Pressing the array button moves the pod configurations into an array. Draws attention to multiplication-as-array.

Table 1: The semiotic potential of TT for learning multiplication

Although not all possible aspects of the semiotic potential of TT have been listed here, we have tried to provide the most important ones. Already, one can point out that these meanings do not all line up well with multiplication-as-repeated-addition, nor with the typical interpretation of multiplication in North America in which the multiplier precedes the multiplication. That said, the set of pods that can be seen in Figure 1b can be read as four groups of three, and thereby interpreted in a multiplication-as-repeated-addition manner, showing that the semiotic potential of TT does not preclude this approach either.

Results

We focus on two teachers in particular, Susan and Leah, who both had opportunities to interact with TT themselves during the two prior meetings and who both shared their experiences of using it in classrooms. From the video transcript of the third meeting, we have identified three episodes that illuminate aspects of the semiotic potential of TT in relation to learning-to-teach multiplication, episodes which arose over the course of the meeting. They are presented here in chronological order.

Equation as balance

The meeting began with Susan showing a video of her kindergarten students playing with TT. After some initial excitement about what the students were
doing, she described how “there was another group [of students] and they knew to say, and I don’t know how or where along their learning they learned this, but they were like two times ten is twenty”. She was surprised that the students could read the equation off, but then she said, “except when it was going the other direction, they weren’t reading it correctly”. Here, she was referring to the ‘reverse order’ of the equation that appears if the pips are placed on the right half of the screen and the pods on the left half, producing ‘20 = 2 x 10’. Sandy suggested that this incorrect reading could be “a good learning opportunity because 20 is the same as 2 x 10. To get that balance maybe.” Leah responded by saying that after discussing the directionality of the equation at the previous meeting, she “did a little pre-test” and found – much to her surprise – that almost all of her students thought that the equal signs means “comes next”. She then said, “Okay, time to unprogramme you”, referring to her focus on helping her students see the equal sign as “equivalency, a balance”.

In this example, the artefact sign in TT (the equation displayed at the top of the screen) prompted a discussion about the directionality of the multiplication equation. In turn, this became a pivot sign for Leah, prompting her to notice a particular feature of her own practice, which was her exclusive use of equations where the result is always at the end. This led to a discussion of the mathematical ‘=’ sign. We note here that Leah’s change in practice did not involve the use of TT with her students, but was instead prompted by her experience of using TT and discussing a particular artefact sign in the teacher meeting. Further, it was Susan’s comment on her children’s reading of the equation that triggered Leah to share this change in practice.

**Multiplication earlier**

Susan repeats her comment about being surprised that her kindergarten students could use the word “times” and wondered where they had picked this up, since she had not used the word at all when inviting them to play with TT.

Susan: And another kid said because two times ten is twenty and he just automatically started counting by ten. Ten twenty thirty forty fifty sixty.

Sean: Mmm.

Susan: That shows me, you know how he got to that, like even though he made two times ten, but he knew.

Sean: He spoke it, he didn’t necessarily make it.

Susan: He spoke it. No, like he, they made it.

Sean: Oh okay.

Susan: He spoke it and then he read the ... ‘cause I always ask, “So how do you know?” And he said because two times ten is twenty and then he started counting by ten.

Leah: Oh yeah, ‘cause kids know.
This conversation was prompted by an artefact sign in TT, namely the appearance of the number symbols and the equation at the top of the screen, which is, in turn, linked to the pip-touches and pod-touches. Susan was at first impressed that the children could read the equation, but then also argued that the kid was not just reading the answer off the screen, but knew something too, perhaps because he was able to continue counting by ten. Susan also described other instances in which the children were noticing and then saying equations out loud like, “three times three is nine” and “two times six is twelve”, and also an occasion when the children said, “let’s try five”, which she took to be more evidence that the children could work with multiplication even if they were only in kindergarten. Leah then said, “I’m really, really starting to rethink this about our curriculum. I actually feel that teaching multiplication is so much easier than […] I feel that they have an innate understanding of it after grade 1”.

The artefact signs in TT seemed to enable the children to express equations out loud and count out multiples, which became, for Leah and Susan, pivot signs of the children’s understanding of multiplication. These pivot signs prompted the teachers to re-think their assumptions about when multiplication should be taught and how difficult the concept of multiplication really was. Perhaps knowing multiplication was not just understanding the idea of two groups of ten, but also being able to produce ‘2 x 10 = 20’ in TT, to count by tens and to use the language of multiplication correctly. In this case, the pivot signs arose from Susan’s recounting of her students using TT, which Leah seemed to find both surprising (“Wow”) and confirming of her own growing sense that students can have a sense of multiplication even before grade two.

**Multiplier and multiplicand**

When discussing a closed website where the teachers and researchers have assembled some resources, including a description of different models for multiplication, Susan commented, “What I’m wondering is how do I, in kindergarten, not teach them bad habits? Not focus on it in the traditional way.” Sandy replied that exposing the students to different models of multiplication was already very helpful. Leah then spoke of her “discomfort originally” with the order of the terms in the equation “not matching groups of”, which she declared she was “now over”. She did note, however, that although she had not formally taught her students multiplication, “they’ll say ‘oh, that’s three groups of two’ even though it says three two times”. Leah then explained how the textbook was always using “groups of” but that TT “shows it the other way”, which “is a good thing. It shouldn’t matter”. When Sean suggested that textbooks want to present the easiest approach in a consistent way, Leah lamented that this “takes away from the commutative property”. Several minutes later, about 3 x 3, she said, “So I’m learning to say three, three times […] forcing me to use different language and there are times when it does matter”.

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This episode demonstrates how Leah reconciled her discomfort around the artefact sign of TT, with respect to the multiplicand preceding the multiplier. This seems to be due, in part, to observations of her students speaking in terms of “groups of”, like the textbook, and hence being flexible about the order. It seems also to be due to her growing appreciation for the importance of the commutative property and perhaps is also linked to an emerging interest in remaining open to the meaning of the equal sign. This caused her to rethink the language she uses for multiplication. Leah’s new use of the ‘multiplicand, multiplier times’ terminology can be viewed as a pivot sign in her mathematical thinking that has been prompted by TT.

Conclusion

With regards to implications of this research for teachers, as demonstrated by the three episodes above, the unfolding of the semiotic potential of TT produced different kinds of changes for the teachers. Shifts in thinking and teaching related to the order of multiplier and multiplicand were provoked by a TT artefact sign and involved a radically new way of conceptualising multiplication. The change in thinking and thinking about teaching related to the equal sign was also provoked by a TT artefact sign, though the focus was more on the importance of helping students develop a better understanding of it. The change in thinking about the appropriate age for teaching multiplication was provoked by observing kindergarten children learn from and come to know using TT, particularly in terms of their unexpected use of certain mathematical discourse.

This is the beginning of a significant project that we believe will have future implications for teachers and teacher education. In the next phase of work, we intend to explore further the semiotic potential of TT when used with tasks specifically designed to highlight, both for primary students and for their teachers, the functional, symbolic and simultaneous model of multiplication afforded by TT. Additionally, we will be studying the use of TT in the classroom by the teachers that are currently part of the project, and we expect that these experiences will trigger different artefact signs based both on what the students notice and on how TT gets incorporated into the teachers’ broader resource system.

References


COUNTING ‘GAMES’ FOR YOUNG CHILDREN: AFFORDANCES AND LIMITATIONS OF TABLET APPLICATIONS

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Abstract

More and more young children are spending time engaging with game-like applications on touch-screen tablets. This paper analyzes fifteen tablet applications in terms of their affordances and limitations when promoting counting objects. Visual, auditory, and gestural aspects are considered. Findings indicated that verbal counting and one-to-one correspondence are given more attention than cardinality. Suggestions are offered for adults and educators wishing to make use of tablet applications when promoting children’s counting competencies.

Keywords: counting, enumerating, tablet applications

Introduction

The motive for this study came about while planning a program for adults (not kindergarten teachers) related to young children’s early number conceptions. As part of the program, we wished to present activities to the participants that can promote early number skills, such as verbal counting and object counting. Taking into account the growing use of tablet applications (apps) in education (e.g., Moyer-Packenham, Salkind and Bolyard, 2008), we also searched for appropriate ‘game’ apps. Needless to say, there are many apps that claim to promote early number skills, and we found ourselves in need of a tool that could help us, and other educators, analyze the mathematical affordances and limitations of such apps. Do the apps promote each of the different competencies involved in object counting, or only some of them?

Several studies have analyzed the use of iPads, tablets, and touch-screen technology in early childhood (ages 4–7) mathematics education. Some studies focused on game apps that are not specifically directed at practicing mathematics skills, but may still afford opportunities to engage with mathematics (Lange and Meaney, 2013). Other studies focused on the use of virtual manipulatives, which may be defined as “computer based renditions of common mathematics manipulatives and tools” (Dorward, 2002, p. 329). Specifically related to counting, Sinclair and SedaghatJou (2013) designed an instrument for the iPad based on the approach that “mathematical thinking is centrally constituted by bodily activity” (p. 2201). In their study, they found a strong relationship between finger use and how children were thinking of numbers.

In this study, we analyze game-like applications that are specifically designed to engage the young child with counting objects. Touch-screen technology offers
users a possible combination of visual, audio, and gestural (e.g., tapping, dragging, and swiping) experiences. Radford et al. (2005) viewed gestures as part of the semiotic means that allow students to objectify knowledge and notice abstract relationships. Dubé and McEwen (2015) found that different touchscreen gestures, such as tapping and dragging, may impact differently on learners’ understanding of mathematical concepts. Taking the above into consideration, this study aims to analyze how the design of an app may promote or limit the learning of counting objects. Specifically, we ask: how might the visual, audio, and gestural possibilities of an app come together to support enumeration?

Enumerating: Principles and complications

Counting objects, also called enumerating, involves several skills. First, there are what Gelman and Gallistel (1978) called the three “how-to-count” principles: the stable-order principle, the one-to-one principle, and the cardinal principle. Implementing the stable-order principle is based on being able to count verbally. This refers to saying the number words in the proper order and knowing the principles and patterns in the number system as coded in one’s natural language (Baroody, 1987). The one-to-one principle involves assigning one count word to each object. Common mistakes related to this principle occur when an object is skipped over, and not counted, or when one object is assigned more than one count number (Fuson, 1988). The third “how-to” principle is cardinality, which involves knowing that when counting objects in a set, the last number mentioned represents the number of objects in that set. A child who has not yet understood this principle, may simply state any number when asked how many objects are in a set, or recount the objects which have just been counted (Fluck and Henderson, 1996).

In addition to the first three principles, there are the two “what-to-count” principles: the abstraction principle, and the order-irrelevance principle (Gelman and Gallistel, 1978). The abstraction principle means that any set of discrete objects can be counted. The order-irrelevance principle includes knowing that one may enumerate the objects in any order (e.g., from right to left, from left to right, etc.) and that enumerating objects in different ways results in the same cardinality. According to Baroody (1984), kindergarten children readily accept that objects may be enumerated in several ways, but have difficulty accepting that cardinality stays the same. It is important to note that children may show knowledge of one principle while violating another principle; for example, erring with regard to the one-to-one correspondence principle, but showing understanding of cardinality (Fuson, 1988).

Several studies have investigated how the layout of the objects may affect children’s success in counting. One study compared children’s ability to count items placed in a row versus items placed in a circle (Tirosh, Tsamir, Barkai and Levenson, 2018). While all children were able to count the items placed in row, only half correctly counted the items when they were placed in a circle. The most common error was over or under-counting by one. In fact, when the items were
placed in a circle, some children did not even attempt to count them. The authors suggested that children may not have experience counting objects that are not arranged in a specific order. Spatial arrangements of objects can also affect ability to subitize (Arp and Fagard 2005). Sarama and Clements (2009) differentiated between perceptual subitizing and conceptual subitizing. Perceptual subitizing refers to recognizing a number without knowingly employing other mental processes, and then naming the number. Conceptual subitizing refers to recognizing a set of objects as two (or more) collections via perceptual subitizing. For example, when shown five dots, a child may say five, because he sees three and two (Sarama and Clements, 2009). Subitizing also supports advanced counting strategies, such as counting on. For example, when shown seven dots, a child may subitize three, and then count, four, five, six, and seven. Finally, Sarama and Clements (2009) point out that “counting out” a set of objects from a larger set (e.g., taking out six candies from a bag containing 30 candies) is more challenging than simply counting six candies set on the table.

Methodology

In this study, we used a document analysis methodology (Bowen, 2009), to analyze fifteen game-like apps that were available for free on Android tablets. We chose to use freely available apps recognizing that most parents (as well as other interested adults) would be more apt to download a free app, than one they would have to buy. In addition, because there are so many more apps available in English than in Hebrew (our native language), we chose two Hebrew apps, and the rest were in English.

Our analysis began by investigating the ways in which apps may support or constrain verbal counting, one-one correspondence, and the cardinality principle. For example, did the app say the number words out loud so that children could hear the counting words in order? A second analysis was directed at the ways in which the objects to be counted were presented. For example, were the objects static or moving? Were they placed in an organized manner? Did the organization of the objects promote the use of subitizing? For each type of analysis, we considered both perceptual, auditory, and motor experiences that may contribute or limit to a child’s enumerating skills. Finally, we note that technological limitations are not analyzed. We acknowledge that little fingers may not always touch an object with enough strength to cause a reaction, or they may tap to the left of an object, instead of on the object itself. We do not consider these limitations when exploring the apps.

Findings

This section presents the findings according to the first three “how-to-count” principles of Gelman and Gallistel (1978). It then analyzes the presentation of objects on the screen, taking into consideration the two “what-to-count” principles.
The stable-order principle

In general, apps that promote verbal counting for non-readers are apps that say out loud the counting numbers, from one to some number, supporting the stable-order principle. In most of the apps surveyed, verbal counting accompanied some action on the part of the player, for example, placing tomatoes in a basket. As the player moves each tomato, the app counts out loud, 1, 2, 3, and so forth. We found six apps that did not say out loud the number words. For example, when placing eight ostriches in a truck, the movement of the ostriches from the zoo to the truck was not accompanied by the sound of verbal counting.

Counting from some number other than one, was possible in some of the apps. For example, when counting eight fish, placing three fingers on three fish, caused the app to say three, and then, when touching one fish at a time, you hear 4, 5, 6, 7, 8. However, if one first touched three fish, and then two, and then three, you would hear 3, 5, 8. Thus, counting by ones may not necessarily be promoted. Finally, none of the apps specifically promoted skip counting. However, as described above, in apps that allow you to touch two or three objects at a time, skip counting could be promoted. None of the apps promoted counting backwards.

One-to-one correspondence

Three of the apps we analyzed did not promote one-to-one correspondence. That is, the player could touch each object on the screen, but there was no feedback from the tablet in response. In all of the other games, attention was given in some way to this principle. In all, except one, the player only needed to tap an object, and the tablet responded, while in that one, the player needed to drag the object from one place on the screen to another.

There were several types of perceptual feedback to touching an object, all of them in some way letting the player know that the object had been touched: the object changed color or brightness; the numeral corresponding to the order in which the object was touched appeared on the object itself; the object moved and changed its place; the object disappeared entirely. Figure 1 shows an example of four trees, which when touched become faded, and a number appears on the tree that was touched, in the order that it was touched. All of these perceptual changes can assist the child in keeping track of what was counted, and what still needs to be counted. They also emphasize that for each touch there is one and only one corresponding reaction. Specifically, if the child touches an object that was already touched (i.e., counted) the object does not change again. In two of the apps, you could touch more than one object at a time, but different fingers had to be used for each object (e.g., I can use three fingers to touch three trees).

As a component of enumeration, one-to-one correspondence means assigning one counting word to one, and only one, object. All apps, except three, responded to touch with a number assignment, either by visually showing the appropriate number symbol (see Figure 1) and/or by saying the number out loud (auditory
feedback) in coordination with the visual change in the object. The other three apps responded to touch by moving the object, but did not assign the movement a corresponding number.

![Figure 1: Counting four trees](image)

A more general notion of one-to-one correspondence, not related to enumeration, entails assigning one item in a set, to one and only one item in a second set. Only one app promoted this general one-to-one correspondence by pairing a sun, for example, with one plant (see Figure 5).

**Cardinality**

In analyzing whether or not the apps specifically promoted cardinality, we first note a difference between apps that requested the user to find, for example, six pigs, and apps that asked, “How many?” (e.g., How many creatures are there?) In the first category, users were able to touch each object and see a change in the object (one-to-one correspondence). In some, they could hear the counting words in order, and with apps that showed number symbols, the last number was sometimes left on the screen, as in Figure 1. That the last number was left on the screen might be considered to promote cardinality. However, there was no follow-up. There was no voice at the end saying, “There are four trees.”

In the second category, we differentiate between two options. The app asks, for example, “How many creatures are there?”, but there is no specific promotion of one-to-one correspondence (as described above). Instead, the child has to tap the correct answer, receiving feedback in the form of a check mark (on one of the apps) or a statement on the screen stating, “correct, there are six pigs.” In only one app, after asking “how many rabbits appear on the screen,” the user touches each rabbit while the tablet counts aloud each touch, and after touching all of the rabbits, sums up by voicing “there are 10 rabbits.” In general, most of the apps we analyzed did not promote the principle of cardinality.

**What-to-count**

The what-to-count principles are related to how objects are presented, which can also be related to the specific counting task. A basic task is requesting the user to count objects, and presenting only those objects to be counted (see Figure 1). A more complex task, is counting-out, requiring the player to count \( n \) objects,
when being presented with more than $n$ objects (see Figure 2). In that case, the extra objects may be identical to those which need to be counted, or different.

![Figure 2: Counting-out five (initial position, three out of five, five)](image)

Related to the abstraction principle, in all of the apps we analyzed, except for one, the items to be counted were identical. That is, if the player was requested to count fish, the fish were exactly the same shape, color, size, and, in the case when the pictures were static, they even faced the same way. In only one app, the items to be counted were not identical, although they were of the same general type (e.g., they were all fish, but different kinds of fish – see Figure 3). Even when the task was counting-out, the objects to be counted were identical. For example, faced with an assortment of vegetables, if the player was asked to count-out three tomatoes, those tomatoes were identical.

Another important aspect of object presentation is the way objects are organized on the screen. This is related to the order-irrelevance principle in that organization might impact on the order in which a player counts the objects. However, regardless of the way objects were presented, all of the apps allowed the player to tap or drag objects in any order. Regarding initial presentation, we found that either objects were arranged in rows, sometimes with an equal number of objects in each row, and sometimes not, or the objects were spread out in no particular way (see Figure 2 where the initial position of the ostriches is spread out). When objects were spread out, they were sometimes arranged in small groups of two or three, promoting subitizing (see Figure 4). As mentioned previously, with some apps, touching the object changed its position on the screen, resulting in a new organization. In all of the apps, when the objects were initially spread out, they ended up in rows (see Figure 2, where the ostriches stand in a row in the truck.). There was only one app where the objects were initially in rows, moved together to a bunch (e.g., all the tomatoes ended up in a basket), and after being touched again, ended up once again in rows (e.g., on the supermarket shelf) (see Figure 5).

![Figure 3: Eight fingers and eight fish](image)

![Figure 4: Subitizing butterflies](image)
Only one app offered the player an opportunity to enumerate the same number of objects using different objects each time. For example, counting four suns, for four plants, and then counting four tomatoes (see Figure 5). A different app presented a number of objects to be counted (in rows), while on the screen a picture of hands holding up the appropriate number of fingers was shown (see Figure 3). Thus, two different representations of the number were present at the same time. In all of the other apps, the number of objects to be counted kept on changing with each new screen, without offering the opportunity to visualize more than one presentation of the number.

![Figure 5: four suns and four plants, become four tomatoes, all in a basket](image)

**Discussion**

This paper began by asking how tablet applications may promote each of the different competencies involved in object counting. In general, findings indicated that different apps have different ways of supporting each competency, along with different limitations. Thus, this section discusses how these findings might help an adult or educator extend the use of an app, offering additional opportunities for learning.

Regarding counting, almost all of the apps we investigated counted out loud in sync with the user’s touch, offering repeated auditory opportunities for hearing the counting words in order. However, in addition to counting by ones, several mathematics curricula recognize the importance of promoting skip counting, for example, counting “2, 4, 6, 8, 10” (Israel National Preschool Mathematics Curriculum [INPMC], 2010). No specific promotion of skip counting was found in the apps analyzed. An adult sitting with a child might suggest that the child specifically touch two objects at time (where the app allows this), affording the child to hear skip counting by twos. There also were no opportunities for counting backwards, an important skill that can support addition and subtraction (Sarama and Clements, 2009). Here too, an adult might find opportunities to count backward with the child when, for example, the tomatoes are taken out of the basket.

Perhaps the most supported competency was one-to-one correspondence. Only one app supported the more general notion of one-to-one correspondence (e.g., by matching one sun to one plant). Taking this into consideration, adults may seek additional ways of supporting this competency, for example, by engaging children with activities such as setting the table, one plate in front of each chair (INPMC, 2010). Another issue, as mentioned in the background, is that children find it
difficult to keep track of what was and was not counted (Fuson, 1988). Most of the apps do this for the child, for example, by shading the object counted. In essence, the app eliminates the problem, and by doing so, limits children’s opportunities for developing tracking schemes of their own. Here, as in Dubé and McEwen’s (2015) study, we take note of the difference between tapping an object and dragging it. Dragging an object from one spot to another, mimics the action a child may take when trying to keep track of physical objects to be counted, and thus might support such a strategy in the future. Of course, this strategy only works when the objects can be moved, either virtually and physically. Keeping track of items is also made easier or more difficult by their organization. An adult might at first choose an app where the objects are arranged neatly in rows, and then move on to apps that present objects spread out. A few of the apps grouped items in such a way as to support subitizing. An adult sitting with a child might point out such groupings, assisting the child to make use of such groupings, and in doing so, promote more advanced counting strategies, such as counting-on. None of the apps specifically arranged objects in a circle, a challenging arrangement for children (Tirosh, Tsamir, Barkai and Levenson, 2018).

Hardly any of the apps supported cardinality. Recall that one app placed the counted items in a bunch (e.g., in a basket – see Figure 5). At this point, not being able to actually see how many items are in the basket, an adult may ask the child, how many items are in the basket. Not being able to actually see how many there are, may strengthen the notion that the last number said when putting them in, represents the amount of items now in the basket. Taking them out and placing them on the shelf may be used as a way of checking the answer to the question of how many.

All of the apps supported the order-irrelevance principle, in that objects could be touch and counted in any order. However, none of them actually promoted it. An adult sitting with a child may focus on this principle by requesting the child to play the same ‘game’ again, but this time guide the child away from counting the objects in the same order. Asking questions, such as, “if we now count from here, how many will there be?” will also support this principle. None of the apps could be said to support the abstraction principle. All of the items to be counted were identical. In addition, presenting only items to be counted may reinforce the erroneous conception that sets must contain items that have a certain explicit common property (Linchevski and Vinner, 1988). Adults should be aware of this limitation, as well.

The analysis carried out in this paper can be used in several ways. First, it can help early childhood educators interested in supporting early counting and enumeration, to choose appropriate apps. Taking into consideration the affordances and limitations of such apps, educators can then plan how to use the apps when interacting with children. The analysis in this paper may also serve as template for analyzing apps with other mathematical aims. The paper
demonstrates the importance of taking into consideration specific mathematical competencies, along with knowledge of mathematics and students (Loewenberg Ball, Thames and Phelps, 2008), in choosing appropriate apps. A next step would be to investigate adults’ interactions with children while engaging with apps.

Acknowledgement: This research was supported by The Israel Science Foundation (grant No. 1631/18).

References
This paper focuses on utilizing regular linear patterns in mathematical education in preprimary and primary education. The ability to recognize and create patterns help us make predictions based on our observations; this is an important skill in math. Understanding patterns help prepare children for learning complex number concepts and mathematical operations. The intent of our research assignment APVV 15-0378 Optimization of teaching materials for mathematics based on analysis of current needs and abilities of primary school pupils is to conduct a field study of the abilities and interests of pupils aged 6 – 8 in mathematics. One of the studied topics are patterns. We were examining the feasibility of the created applets for children aged 5-7. Do linear patterns still hold interest for pupils in primary education?

Keywords: applet, patterns, primary pupils, preschool children, observation, design based research (DBR)

Introduction

Within the cognitive process, children obtain knowledge, experience, they realize the world surrounding them. We find patterns in math, but we also find patterns in nature, art, music and literature. The interconnection of art and mathematics is motivating for pupils, looking for patterns in traditional folk embroidery brings...
creativity into the classes (Gunčaga and Zentko, 2016; Hunter and Miller, 2017), and facilitates mathematic learning even in problematic pupils.

Patterns provide a sense of order in what might otherwise appear chaotic. Researchers have found that understanding and being able to identify recurring patterns allows us to make educated guesses, assumptions, and hypotheses; it helps us develop important skills of critical thinking and logic. The knowledge and understanding of patterns can be transferred into all curriculum areas and open many doors where this knowledge can be applied.

**Theoretical basis**

In this paper, we will concentrate solely on repeating linear patterns. A linear pattern is defined as any series of elements ordered by a core unit of a repeating pattern. The pattern holds information about the series’ construction process. Every element can be assigned a position in the series based on information from the preceding elements and enables placing the succeeding element (Partová and Žilková, 2009).

Patterns are very important for the development of mathematic competences, discovering rules, understanding numeric series. Mathematician L. Steen (1988) „referred to mathematic as the science of patterns – patterns in number and space. The theory of mathematics, according to Steen, is built on relations among patterns and on applications derived from the fit between pattern and observation. “ (Clements and Samara, 2014, p. 215).

Preschool children encounter linear patterns fairly often. They thread tiles on strips, insert pegs in a pegboard, build towers alternating one- and two-cube floors, color a striped pattern with alternating colors and engage in many other manipulative activities. These activities are useful, but it is essential to prepare tasks with increasing difficulty, something that teachers seldom do. Several authors (Clements and Samara, 2014; Skoumpourdi, 2016; Orton et al., 2005) agree that children need to develop these specific mathematic skills by creating linear pattern with as many different elements as possible. Patterning is the search for mathematical regularities and structures. Patterning is more than a content area - it is a process, a domain of study, and a habit of mind. (Clements and Samara, 2014)

In tasks pertaining to linear patterns we observe the development of the ability to identify, complement, reproduce, extend, describe and create patterns in different structures, constructed with a variety of materials. We can classify patterns according to several criteria (Skoumpourdi, 2017), which can affect the pattern’s difficulty:

- The type – identification the core of a pattern, reproduction (copy, duplicate) a pattern, completion of a pattern and creation of a pattern.
- The structure – is describing the core unit of the pattern (ab, abb, aab, abc, …),
• The material – is related to the kind of the means used for constructing the pattern and way it is presented – manipulatives, iconic representations, virtual representations.

• The arrangement – linear (horizontal, vertical), path arrangement and circular.

Patterning is included already in preprimary education. Slovakia is no exception, patterns have always constituted the mathematical curriculum to an extent, but no special attention was paid to them. Patterns were often put together with sorting, ordering and assigning, resulting in the teachers’ common interchanging of ordering and patterning. In the current State Education Program for Preprimary Education (2016, p. 47), in the Logic section, the child is expected to „create (draw) a simple sequence of objects based on a given pattern (up to 6 objects) or rule. The child is able to continue an existing sequence of objects or pictures. The objects may differ altogether, or just by color or size. The child discovers and simply describes the sequence’s rule/pattern.“ In kindergarten, children engage in a sufficient number of manipulative activities, mostly continuing in the ab or abc pattern, but the activities are not of an increasing difficulty, intervention is systematic, because teachers don’t understand the impact of such activities on the development of mathematic thinking, they do not have access to a sufficient number of teaching resources and methodological guidelines are missing altogether. Teaching resources for preprimary education are neither funded nor controlled by state, resulting in a vast number of resources of debatable quality. All of them, without fail, understand patterns as an opportunity to learn colors or shapes and coloring.

While patterns have been included in preprimary education for a long time, they were included in primary education only in the 2008 reform. Currently, we can find patterns mentioned in the mathematic curriculum in first to third grade of primary school in the State educational program for primary education – Mathematic (2009): *A pupil should be able to find several ways of ordering objects, signs, symbols; identify and describe the rule of an existing series of numbers, signs, symbols and add elements to the series based on the identified rule.*

According to the constructivist paradigm teachers should be able to discern, identify the level of a child’s development and formulate tasks accordingly, to achieve the zone of proximal development. A 3-year-old child is able to continue the ab patterns, provided the core unit of patterning is repeated sufficiently (3-5 times). The child is, however, not able to identify the core unit and cannot continue in an advanced pattern, while there may be exceptions. Klein and Starkey (2004, in Skoumpourdi, 2016) also mention problems in analyzing the core unit of a repeating pattern by 4-year-old children.

Only little attention is paid to working with patterns in primary education. In Slovak textbooks the development of functional thinking using linear patterns can only be found in those written by prof. Hejný et al. and translated from czech
language. We often encounter the opinion, that it is no longer necessary to work with patterns in primary level education, pupils already know it, it is easy for them. But examples of using patterns within interdisciplinary relationships (Gunčaga and Zentko, 2016) as well algebraic development (Hunter and Miller, 2017) are proof, that patterns deserve a broader attention. Teachers don’t recognize the importance of patterns, neither do they take into account other types of patterns – cyclic and growing patterns, number patterns.

**Research**

Our project *Optimization of teaching materials for mathematics based on analysis of current needs and abilities of primary school pupils* (APVV – Scientific Grant Agency of the Ministry of Education of the Slovak Republic no. 15-0378) aims to determine the abilities and needs of younger pupils and based on the analysis of the results of the research, create three types of instructional materials for the students in surveyed groups (printed textbooks, worksheets published on the website a dynamic worksheets (applets) posted on the website. The research as a whole is devised as a design based research (DBR), aiming to determine the relations between the underlying educational theory, suggested means of intervention and practice (David 2007, in Partová and Žilková, 2019). Design based research is based on cyclic repeating of intervention in the process of education targeted on the development and alignment of tools and methods utilized in the teaching/learning process. Our research builds on the research of Partová and Žilková (2009, 2010, 2019).

**Research design**

The aim of our partial research was to examine the feasibility of applets concerning with linear patterns for children aged 5 – 7, observe the children’s work with said applets, their choice of tasks, level of success in solving the tasks and interest shown. All recorded results will be used to improve the applets or for the removal of revealed insufficiencies or faults.

The research sample consisted of children of preschool and younger school age (N=75), aged from 3 years and 5 months up to 6 years and 11 months, girls (40) and boys (35), most of them attending kindergarten or first grade of elementary school, living in northern parts of Slovakia. The average age of the children was 5 years and 1 month. The research was conducted at the end of 2018.

The study was conducted with each child individually as a semi-structured interview, using the practical experience method the children were able to work with and experimentally verify the applets. The whole process was recorded on video and described by field records. A record sheet describing in detail the conduct and success level of every task was created for each child.

Data for our research were obtained with help of a group of teachers, who were willing to work with the children. The teacher’s/researcher’s input consisted of
providing a tablet for the child, launching the applet in question and letting the child interact with it. The teachers were instructed not to interfere, unless the child was not able to continue or identify the task at hand. The teacher was also instructed to motivate the child to comment its’ actions.

Our task was to evaluate the video recordings. We were observing the child’s work depending on age and gender – choice of objects, motivation, strategy used, ways of finding and correcting a mistake, level of difficulty achieved.

**Research tools**

We were studying applets aimed at linear patterns, consisting of 5 types of tasks (Figure 1). The mathematical educational applet from authors Žilková and Partová is available at [www.delmat.org](http://www.delmat.org).

The first and most used type is „finish the linear pattern“. The following three concern with correcting the mistake in a linear pattern by omitting, exchanging and inserting a picture. The fifth applet for development of linear patterns tasks the respondent with completing the missing part of the pattern, which may or may not be identical to the core unit of patterning.

![Figure 1: Selection of linear pattern applets by type](image)

The child’s task is to continue the pattern by adding elements one by one or choosing a group of elements to add from the selection, or to correct a mistake in the pattern by omitting, changing or adding an element. Each type of applet is customizable by options: difficulty, direction, type, property, selection (Figure 2). Difficulty is determined by the length of the core unit built with 2 items. Difficulty 1 consists of patterns with \(ab\) core; difficulty 2 contains cores \(abb, aab, aba\); difficulty 3 core \(aabb, bbab, abba, baab\); difficulty 4 cores \(abaa, aaba, babb, bbab\). Property characterizes elements in the pattern by color, shape and size. Selection sets the elements' style – 3D pictures, 2D pictures, 2D shapes and 3D objects. By appropriately setting the options the child increases or decreases the difficulty of the task.
Use of information and communication technologies in the education process is still very popular in Slovakia, in all levels and types of education. Kindergartens use primarily interactive blackboards and digital toys. Primary schools (first through fourth grade) use also tablets and PCs/laptops. The selected sample of children working with these applets were mostly attending kindergarten, most of them worked with the applets in home environments or on their own tablets on kindergarten grounds.

Results and discussion

The research results indicate several interesting findings, but also confirmed our expectations.

During the study, we found out that children manifest higher engagement in patterning with 2D or 3D pictures that with planar shapes or objects. For example, pictures of cars motivated boys to further activity, they chose higher difficulty by themselves, focused and worked longer. The girls preferred other objects, but generally inclined to real objects (e.g. fruit) as well. A great advantage of the applets is the feedback they provide to the child. The applets are designed so that when the child picks the wrong element or tries to place it in a wrong place, the element is returned back to the selection, so the child can’t make a mistake in continuing a pattern. Children learn very quickly that with such a feedback they can solve tasks by trial and error. Within our research sample, children aged 5 and higher quickly abandon the trial and error approach, as it is very time consuming. They realize it is far more effective to think and identify the core unit of patterning and move the picture into the right position without trying which element fits to the right place. In so doing they achieve success sooner, which is another advantage of this approach.

We found very interesting, that children aged 5-6 were already able to simplify the core unit of patterning. Patterning in \textit{abaa, aaba, baaa} can be seen in extended patterns as \textit{aaab}. 5-year-old girl Marianna started to chant and point to the pictures in pattern: “green, green, brown, green, green, brown, ...” then stopped for a while and continued “green, green, green, brown”. Then the teacher asked Marianna what will follow, and Marianna answered “3 greens”.

In our opinion it is suitable to encourage the children to discover themselves one of the strategies of pattern continuation by asking complementary questions. Threlfall (in Orton, 2005) presents, that children solve these tasks using one of
two strategies. They may set up a chant or refer to the unit of repeat. Both approaches enable the prediction of subsequent items in the sequence, but in different ways. The unit of repetition can be detected in the rhythm of the chant, in the other approach there is evidence of qualitatively different level of awareness, with completely different implications for mathematical understanding.

Also our assumption that applets can be used in development of linear patterns only by children aged 4 and higher, was confirmed. Younger children (3-4) experienced problems with drag-and-drop manipulation on the touchscreen. They were also unable to identify and complete patterns of a higher difficulty than \( ab \). It is, however, needed to hone the sense for rhythm and repetition at this age as well, utilizing manipulative activities.

Even if the aim of our research was not monitoring the teachers, during the video recording analyses we repeatedly confirmed, that a very common shortcoming in Slovakia is the very directive approach of a teacher to the teaching/learning process. Many teachers want to have the process firmly under control and choose assignments, which the children fulfill, or provide an adequately strong support and help in solving them.

In the experimental verification of applets, the teachers picked first the lowest difficulty and alternating the pattern elements by color. Orton (2005) recommends the individual elements to differ in color and size, color and shape, size and shape, but these options are not currently supported by the applets.

Most of the teachers approached the applet verification randomly. Around a quarter of teachers systematically increased the difficulty, but only changed the pictures, and less than a quarter of the teachers systematically increased the difficulty while also changing the elements’ properties (color, size, shape) on every difficulty level.

Another shortcoming was the inability to process errors and mistakes. Teachers didn’t know how to advise a child in looking for an error in the pattern. On one hand we understand this to be a specific feature of our research sample, mostly consisting of beginning teachers with little practice. On the other hand also a beginning teacher should be able to intervene and advise the children so that they can discover the mistake and correct it themselves.

Conclusions

In modeling assignments and tasks for children of preschool children and pupils, it is necessary to observe the gradation of tasks based on the increase of the core unit of pattern elements. Within this matter, it is imperative to create quality methodological guidelines for teachers in kindergarten and for primary teachers as well as worksheets for children and pupils. It is also our intention to create and publish materials available for printing and designed for frontal work with the children on the aforementioned website.
Based on our present experience and the results of this study we recommend to a bigger attention to be paid to at least linear patterning in the future teachers’ curriculum. It is not enough to rely just on children to figure it out for themselves, on patterning to be “easy”. At the same time, we know that tasks and assignments with linear patterns bring children joy from learning and practicing mathematics, while also developing their specific mathematic skills. This joy will also influence the pupils’ opinions on and stands towards mathematics in the future.

In pre-gradual education of future teachers, we recommend to focus on modelling various assignments of various types as described in the theoretical part of this paper. It is also important to teach future teachers to only be the partners/helpers of the students, who obtain knowledge by their own experiences. This is arguably the most difficult challenge presented to teachers in Slovakia – to abandon the directive style of teaching, very much based on memorizing, copying and transmissive teaching.

Acknowledgement: This study was supported by APVV 15-0378 OPTIMAT – Optimization of teaching materials for mathematics based on analysis of current needs and abilities of primary school pupils and GAPF 2/20/2018 and 2/34/2018.

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**STRIKE A POSE: THE IMPACT OF PROBLEM-POsing ON ELEMENTARY STUDENTS’ MATHEMATICAL ATTITUDES AND ACHIEVEMENT**

Danielle Bevan, Ashley M. Williams and Mary M. Capraro

Abstract

Students do not always possess positive feelings toward mathematics. It is therefore imperative to foster students’ motivation and interest toward learning mathematics, and a strategy for increasing interest and motivation is through creative thinking activities, such as problem posing. In spring 2018, elementary students (N = 53) in 2nd and 4th grades were assigned to a problem-posing intervention. A quasi-experimental design was used to determine the relationship between increased mathematical attitude in problem-posing activities and mathematics achievement. Students were administered an attitude survey and achievement test at the beginning and ending of the intervention period (10 weeks). Students whose attitudes increased by more than three-points (n = 11) were compared to students whose attitudes decreased or only had between a one- and two-point increase (n = 42) to determine the relationship between increased attitude and mathematical achievement. Results showed a positive, statistically significant relationship (p < 0.001) between increased mathematical attitude and mathematical achievement in students who were involved in problem-posing activities. Results suggest that allowing for creative thinking activities leads to both positive attitudes toward mathematics and increased mathematics achievement among students.

**Keywords:** creativity, mathematics, problem-posing, problem-solving, elementary students

Students do not always associate mathematics with positive feelings. Ensuring that students are receptive to and engaged in mathematics classrooms, therefore,
necessitates that mathematics teachers know the content as well as the curriculum goals, all of which are centered on ensuring student engagement and learning (National Council of Teachers of Mathematics [NCTM], 2000). Students who do not have a positive outlook on their own abilities in mathematics or who lack positive feelings toward the subject often lose interest and become disengaged in the subject. Incorporating creative activities in mathematics classrooms has been shown to increase students’ positive attitudes toward mathematics (Akay and Boz, 2017; Candiasa, Santiyadnya and Sunu, 2018; Guvercin, Cilavdaroglu and Savas, 2014; Seechaliao, 2017; Walkington, 2017; Walkington and Bernacki, 2015). In addition to improving students’ positive attitudes toward mathematics, incorporating activities that foster students’ creativity in mathematics has been found to enhance their problem-solving and problem-posing skills (Nadjafikhah, Yaftian and Bakhshalizadeh, 2012). A general definition of creativity in education is the act of developing novel ideas that benefit one’s education (Seechaliao, 2017). One method that has been shown to influence students’ attitudes toward mathematics is the use of instructional strategies that allow for and necessitate student creativity.

**Instructional Strategies to Foster Creative Thinking**

Various instructional strategies can be used to help foster creativity in the classroom. Seechaliao (2017) identified eight instructional strategies that help develop creative thinking. The first is brainstorming; during this time, students are collaborating with classmates, receiving frequent feedback from the teacher, and developing critical-thinking skills. The second strategy is thinking outside of the box; during this strategy, the teacher poses questions to help students think of new ideas and more flexibly. The third strategy involves providing students with questions to aide in their creative thinking, and the fourth involves providing students with feedback and reinforcement. The reinforcement should be positive to help create a welcoming environment in which students feel comfortable with taking chances that might lead to mistakes and with developing creative and critical-thinking skills. Fifth, teachers can use games to make learning more interesting for students, and sixth, teachers can use certain teaching techniques periodically; these techniques include “lecturing, demonstration, small group, field trips, induction and deduction” (p. 205). Seventh, as a creative teacher, you need to help students become motivated to want to learn. Motivation is both intrinsic and extrinsic, but a teacher can help students gain confidence by helping build motivation to want to successfully acquire mathematics knowledge. Lastly, the eighth strategy is to keep up with current technology and incorporate it in the classroom. Technology has been proven to help with mathematics achievement (Burns and Hamm, 2011; Chang, Wu, Weng and Sung, 2011; Clements, Battista and Sarama, 2001; Khasawneh, 2009; Olkun, 2003; Topalli and Cagiltay, 2018) and provides new contexts and tools for creative output (Henriksen, Mishra and Fisser, 2016; Koehler et al., 2011). Overall, the instructional strategies provide...
students more autonomy in their learning and provide greater room for the creative exploration of academic topics and concepts.

The eight instructional strategies listed above can be incorporated into classes within any discipline in order to foster students’ creativity. However, one mathematics-specific strategy that allows students to express their mathematical knowledge and creativity is problem posing. The phases for problem posing, adapted from Polya’s (1945) four-phases to problem solving, are pose problem → plan → carry out → look back. Creativity is a process and has phases similar to the design of problem posing (Leung, 1993). When a teacher incorporates creative thinking into problem-posing classroom activities, students have a better chance of becoming more successful in mathematics leading to improved interest, attitudes, and achievement.

**Problem Posing, Creativity, and Students’ Attitudes**

The use of instructional strategies, such as problem posing, is beneficial as teachers try to foster students’ creativity, interest in mathematics, and overall mathematics achievement. “A problem-posing situation is referred to as semi-structured when students are given an open situation and are invited to explore the structure of that situation and to complete it by applying knowledge, skills, concepts, and relationships from their previous mathematical experiences” (Van Harpen and Sriraman, 2012, p. 205). As a student progresses through school, his or her interest in mathematics tends to decrease; however, allowing creative thought to occur through activities such as problem posing can increase the student’s interest and involvement in learning (Walkington and Bernacki, 2015). Problem posing requires direct and active participation of students in the process of learning mathematics, which leads to greater interest in the subject (Candiasa et al., 2018) and fosters positive outlooks and attitudes toward mathematics (Akay and Boz, 2010). Another benefit of problem posing is that the activity can reveal students’ mathematics knowledge and any existing misconceptions they hold (Kilic, 2017). Correctly posing problems is key to understanding mathematics concepts, and the types of problems students pose reflect students’ mathematical knowledge (Chang et al., 2011; Toluk-Ucar, 2009). Prior research studies (Chang et al., 2011; Toluk-Ucar, 2009) and a meta-analytic analysis (Rosli, Capraro and Capraro, 2014) of problem posing have shown a correlation between student participation in creative problem posing and more positive student attitudes toward and interest in mathematics. For instance, in one previous research study, students were given the opportunity to use puzzle games to enhance their problem-posing skills and interest in mathematics; adding the puzzle activity to the students’ mathematics instruction enhanced their interest in mathematics because they were actively involved in their learning (Candiasa et al., 2018). In addition, researchers conducted a meta-analysis on problem posing and attitudes toward mathematics and found a Hedge’s g of 0.76, indicating that problem
posing had a positive effect on students’ mathematics attitudes (Rosli et al., 2014). Creative thinking opportunities, like those found in problem-posing activities, allow for more student ownership in learning, leading to meaningful changes in students’ mathematics attitudes and achievement.

Utilizing problem-posing activities in the mathematics classroom allows teachers to apply strategies that foster creative thinking, improving students’ attitudes and achievement. Teachers can encourage creativity in the mathematics classroom and test students’ knowledge and understanding of mathematical concepts through problem posing and giving students the opportunity to write their own word problems. In addition, students can examine and reflect upon the problems they have created to determine if their problems are realistic and solvable. In the current study, researchers aimed to investigate how problem posing can increase students’ positive attitudes toward and achievement in mathematics and the relationship between the two. As a guide, the researchers used the following research question: What is the relationship between attitudes toward mathematics and the mathematics achievement of students who were exposed to problem-posing activities?

Method

Researchers used a quasi-experimental research design to investigate the relationship between elementary students’ attitudes toward mathematics and their mathematical achievement when exposed to problem-posing strategies. The research team conducted an intervention at four public schools in the southwestern United States during the Spring 2018 semester. Pre-service teachers who were enrolled in a university senior-level course administered the intervention weekly at the public schools for a ten-week period. The pre-service teachers were assigned to a classroom by the university, and the research team had no input or knowledge of pre-service teacher placement. Before administering each of the interventions, pre-service teachers met weekly to outline and discuss the upcoming lesson and explanations on how to deliver the lessons with fidelity for both the problem-posing and problem-solving groups. Quantitative data were collected to analyze student attitudes toward mathematics and mathematics achievement.

Instrument

Determining attitude toward mathematics in elementary students is often thought to be difficult. Researchers developed a short three-question survey that was administered to all participants which was administered to the elementary students prior to the first intervention lesson. At the end of the ten intervention lessons, the same survey was given as a post-survey. The pre-/post-survey consisted of a Likert-scale that was used to assess students’ attitudes toward mathematics. The short survey prompted students to choose which emotion emoji best described their feelings (see Figure 1). Emoji survey responses were replaced with numbers
1 to 6, with 1 being the lowest (sad emoji) and 6 being the highest (heart eyed emoji), with a max score of 18.

In addition to the survey, students also took an identical pre-/post- achievement test. This test consisted of four problem-solving questions and two problem-posing questions. The problem-solving questions were worth 3 points each, and the problem-posing questions were 6 points each. Thus, the achievement test had a max possible score of 24 points. The test was evaluated using a rubric and graded by two researchers to account for inter-rater reliability. Once the pre-/post-survey and achievement tests were completed, the total scores and differences between pre-/post-survey and achievement test scores were calculated.

How do you feel about mathematics? (Circle One)

![Emojis](image_url)

Figure 1: Example from pre-/post-test survey

**Participants**

Participants (ages 7 – 10) in this study were elementary students from second and fourth grades ($N = 87$) from four elementary schools. The average number of students categorized under low socioeconomic status based on statistics from the four schools was approximately 62.1% (Texas Education Agency, 2018). Students in the self-contained classes were randomly sorted into problem-solving (PS) or problem-posing (PP) groups by elementary school and preservice teacher placement, with two schools assigned to PS and two to PP. Student data were collected from the participants with parental consent and student assent. The participants were assigned to the problem-solving ($n = 34$) and problem-posing ($n = 53$) groups based on the school they attended. For the purposes of this study, students in the PP group were examined to determine the relationship between attitude toward mathematics and mathematical achievement when using the creative teaching technique of problem posing as an intervention.

Researchers in the present study then wanted to investigate more closely the relationship between students’ improved attitudes and their achievement scores. To do so, a more focused sample was created that was limited to the students whose attitudes toward mathematics improved the most. When determining the new population size, and if any scores suppressed or affected the outcomes, difference of scores were ranked from least amount of change to most amount of change. First, any students who scored a 17 or 18 (max 18) on the pre-survey were removed, yielding a population ($n = 34$). Students who demonstrated a positive improvement in attitude from pre-survey to post-survey of at least three points were selected as the sample ($n = 11$).
**Intervention**

The intervention consisted of 10 review lessons/activities on specific mathematical concepts after which the students were asked to apply their knowledge of the concept through either PS (solving word problems) or PP (posing word problems). In the PP group, students were asked to create/pose their own word problems based on the concept that was reviewed during the lessons, often using pictures, different manipulatives, or various mathematical expressions. The lessons were designed by the researchers to align with state standards and the scope and sequence at each grade level.

**Data Analysis**

The attitude survey and achievement test results were analyzed to answer the research question investigating the relationship between mathematical attitude and mathematical achievement in students who were engaged in a semester-long problem-posing intervention. Stata 15.1 (STATACORP LLC, College Station, TX, USA) was used for the analysis. Descriptive statistics, 95% confidence intervals, and Hedge’s $g$ effect size were reported to explain the relationship between attitude and achievement. A paired sample $t$-test was used to calculate the mean differences between attitude and achievement.

**Results**

Students who had an improvement of at least three (3) points on the attitude survey were compared to the remaining students in the sample. The paired sample $t$-test results confirmed students whose attitudes increased the most had a statistically significant ($p < 0.001$) increase in their mathematical achievement compared to the PP students with lower attitude improvement scores. The means and standard deviations of achievement post-test for both groups were computed and used to calculate the Hedge’s $g$ effect size displayed in Table 1.

<table>
<thead>
<tr>
<th>Participants ($N = 35$)</th>
<th>Positive Change in Attitude ($n = 11$)</th>
<th>No or Little Change in Attitude ($n = 24$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>20.18</td>
<td>18.88</td>
</tr>
<tr>
<td>SD</td>
<td>3.21</td>
<td>3.94</td>
</tr>
<tr>
<td>$p$</td>
<td>$p &lt; 0.001$</td>
<td></td>
</tr>
<tr>
<td>Hedge’s $g$</td>
<td>$g = 0.34$</td>
<td></td>
</tr>
<tr>
<td>95% CI</td>
<td>[-0.36, 1.04]</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Descriptive Statistics for Positive and No or Little Change in Attitude

The Hedge’s $g$ effect size $g = 0.34$ indicates a positive relationship between attitude and achievement. The students who had the highest increase in attitudes...
also scored statistically significantly higher on their achievement test, which was used to measure their problem-solving and posing skills.

**Conclusion**

Students can express their knowledge and creativity in the mathematics classroom through problem posing. Problem-posing activities can improve attitudes toward mathematics and mathematics achievement (Sugito, Susilowati, Hartono and Supartono, 2017; Sung, Hwang and Chang, 2016). This study adds to the literature by demonstrating that elementary students who were consistently engaged over a semester with problem-posing activities showed some positive improvement in both their attitude toward and achievement in mathematics. Improving attitudes toward mathematics and achievement in mathematics classrooms will always be a goal of mathematics educators and researchers. Results from this study provide evidence that problem-posing activities can improve students’ attitudes. Furthermore, those participants whose attitudes improved the most also had higher achievement scores in the areas of problem solving and posing, which indicates that there exists a relationship between students’ attitudes towards mathematics and mathematics achievement.

Problem-posing activities in the classroom encourage students to have more ownership in their learning and have been shown to increase their interest in mathematics. When students have positive attitudes toward mathematics, they are more likely to be successful in understanding mathematical concepts (Seechaliao, 2017). Teachers who allow for and foster creative thinking through activities such as problem posing can help lead their students to gain a better mathematics foundation. Problem-posing activities afford students the opportunity to be creative in mathematics classrooms, increasing students’ positive attitudes and achievement. Teachers can help lead students to take more ownership and be more creative in their learning.

Teachers play an important role in the dissemination of mathematics knowledge to future generations. In order to be better equipped to utilize problem-posing and encourage creativity in the classroom, teachers could attend professional developments offered through their schools or local universities. In addition, to encourage future mathematics teachers, the incorporation of problem-posing can be introduced and taught in teacher education courses at universities. It is important for teachers to be adequately prepared to integrate instructional strategies that foster creativity in mathematics classrooms in order to better demonstrate and encourage the use of creative mathematical thinking.

**References**


The Homework in Different Approaches – Scheme-orientated, Montessori and Undesignated Approach in the Czech Republic

Miroslava Brožová

Abstract

This article reports on assigning homework in three different elementary level teaching methods for mathematics in the Czech Republic: the Scheme-oriented approach, the Montessori method, and the undesignated mainstream approach. The aim of this study was to find out if the teachers, using the different teaching methods, assign homework in teaching mathematics; for what purposes they assign it; and if they find it effective. Analysis of interviews with six teachers showed that assigning homework partly depends on the teaching method. With the undesignated mainstream approach, homework is assigned daily. In the Montessori method and Scheme oriented approach teachers see them as the opportunity for children to finish their work or to look back and correct their own mistakes.

Keywords: homework, elementary, mathematics, scheme-oriented, Montessori

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Introduction

With increasing demands on the quality of education in the Czech Republic, homework occasionally becomes a source of conflict between home and school. Some intense discussions appeared during recent years in academic publications (e.g. Puhrová (2018); Holte (2016); Costa at al. (2016); and especially in commercial press).

All participants in the educational process (students, parents and teachers) seem to have some concerns. Parents usually ask if the assignments can be shorter or longer, harder or easier, or can be more ambiguous (Baumgartner, Donahue and Bryan, 1998; Kralovec and Buell, 2003;). Teachers need significant support from parents, as well as training in how to construct good assignments, and time to prepare effective assignments (Farkas, Johnson and Duffet, 1999). Students protest about the time that they need to spend working at home (Coutts, 2004; Kralovec and Buell, 2003). Many students consider homework the chief source of stress in their lives (Kouzma and Kennedy, 2002).

Kralovec and Buell (2003) focused their research on the context of the lives of students, families, and communities. They found that homework often disrupts family life and interferes with what parents want to teach their children.

Math homework is usually assigned daily. It is chosen from problem sets following the section of the topic that was addressed that day and the entire class can get the same assignment (Deubel, 2007).

The main purpose of this study is to describe specific aspects of three teaching approaches in which the author of this paper is interested.

The aim of the study is to answer the following questions:
1. Do the teachers, using the Scheme oriented approach, the Montessori method, and the undesignated approach, assign homework in teaching mathematics?
2. For what purposes do the teachers assign homework? Do the teachers evaluate homework as effective? Which positive or negative effects does it have/could it have?

Theoretical Framework

Homework was defined by Cooper (1989a) as: “Tasks assigned to students by school teachers that are meant to be carried out during nonschool hours,” (Cooper, 1989a, p. 7). Pezdek et al. described homework as the major vehicle through which parents help their children with school subjects (Pezdek, Berry and Renno, 2002). Cooper suggested that students may complete homework assignments during study hall, library time, or even during subsequent classes. Jursová (2011) considered homework as work for the home environment. She listed activities including home parent-to-child teaching, help with assigned school tasks (projects, preparation of aids, delivery of messages from school, etc.), but also
extracurricular activities dedicated to support the child’s learning such as discussions over books, encyclopedias, the child’s hobbies, playing a musical instrument, or joint conversations between the parent and child (Jursová, 2011).

Holte (2016) discussed the point and effects of homework assigned to lower primary school children. Desforges and Abouchaar (2003) claim that the involvement of parents and their support has not only a strong impact on the development of the child during the first school years, but also creates a positive influence on their school results in adolescence (Puhrová, 2018). Opinions about the impact of homework diverge. Even Kralovec and Buell (2003) indicate Cooper as a supporter of homework, pointing out that Cooper conceded in his work (Cooper, 2004) that homework does not improve academic achievement for elementary students. Cooper (1989a) reviewed nearly 120 US educational studies on the effect of homework on student achievement. Studies comparing the achievement of students with and without homework tend to find no association in primary education.

Although few studies have examined the effect of homework time specifically on mathematics achievement, several studies have examined the effect of homework time on scholastic achievement generally.

The literature finds that variations in homework can be classified according to its (a) amount, (b) skill area, (c) purpose, (d) the degree of choice (e) completion deadlines, (f) degree of individualization, and (g) social context. The purposes of giving homework also vary. Students should practice or review what they have learned in the class; parent and child should communicate; the teacher fulfills expectations from the head of school; or the teacher wants to punish students (Cooper et al., 2006). Homework assignments rarely reflect a single purpose.

Some assignments are meant for the student to complete independent of other people. Assisted homework explicitly calls for the involvement of another person: a parent, or perhaps a sibling or friend. Still other assignments involve groups of students working cooperatively to produce a single product.

Fynewever (2008) studied the electronic ways of setting the homework. Students appreciate the usefulness of both paper-based and web-based homework as a preparation for quizzes and examinations. Advantages unique to a typical use of web-based homework include instantaneous feedback and ability to resubmit assignments.

In 2009, Grønmo and Onstad presented TIMSS results showing that homework was corrected and given feedback by teachers to only a small extent. These authors argued that if homework is not corrected or connected to what is going on in the classroom it might leave the impression that homework is just something that should be done and not something from which to learn (Grønmo and Onstad, 2009). They estimated the effect of homework for 16 OECD countries using data from the Trends in International Mathematics and Science Study (TIMSS) in 2007.
for nine-year old students. Results show that the volume of homework has increased over time (Valdermo, 2014).

In the Czech Republic, teachers are free to choose whether or not to use homework with their students. Numbers show that more than 50% of students in fourth grade bring homework from half of the lessons. The effect of assigning homework is significantly positive for the group of students with the most books at home. The effect of homework is on average more than twice as large for girls than for boys (Falch and Ronning, 2012).

**Montessori Approach**

Individualized pacing and instruction are the features that pronouncedly characterize high-quality Montessori programs (Diamond and Lee, 2011). In such an environment, homework matches each student’s need whenever possible; trying to ensure that practice is meaningful for everyone.

Maria Montessori did not want to dictate the work of the child at school or at home. The children may choose in-class mini-workshops that will help them clarify areas of confusion. The children may voluntarily choose to continue their “work” at home or select homework assignments as an extension of their educational exploration. A task that one group completes in one day may become homework for a less-advanced group within the next few days.

Instead of traditionally set homework Montessori suggested constructive activities that can represent homework. The activities should be adapted to the pupil’s interests and abilities, provide challenges and opportunities for mastery, and include exciting, investigative work. Children are encouraged to spend time, for example, figuring total price at the grocery store, keeping statistics, graphing when they go to bed, and how many pages they read or how far they walk each day. They can pair their socks and count by twos, measure things and calculate their surface area, divide a pizza into equal pieces, or invent patterns for curve stitching.

“Montessori claimed that she had identified resources in children which teachers had not noticed. Perhaps some of these resources will become more visible if homework practices are based on children’s perspectives.” (Holte, 2016, p.30).

**Scheme-Oriented Approach**

The Scheme-oriented approach is described as a “coherent approach enabling children to discover mathematics by themselves and with enjoyment” (www.h-mat.cz). The teacher in this approach presents adequate problems and organises discussions of the students in such a way that it proceeds to the required knowledge. There are three additional requirements placed: connection to a pupil’s life experience; a long-term nature, which means that is useable for pupils of different ages; and a differentiated nature which enables them to pose problems to cater to the needs of individual pupils (Hejný, 2012).
The edition of textbooks for elementary schools used for teaching with this approach has been approved by the Czech Ministry of Education in 2007.

It was designed for children to learn primarily at school; therefore, homework can be voluntary, and discussion with those who did it follows. It should help children take responsibility for their learning and planning. One of the parents says that children come home with new discoveries and interesting tasks, and are enthusiastic to explain them to parents. (www.h-mat.cz)

Each child discovers and achieves the results individually or during the class discussion. Students themselves can determine the pace, direction, and extent of their work and later can also choose the difficulty of their homework.

**Undesignated Mainstream Approach**

This approach is used in most of the schools in the Czech Republic on a long-term basis. Teachers do not use any one specific method to teach mathematics. They usually combine several approaches and concepts, using, for instance, projects and game-based activities. It is the teacher’s responsibility to teach students all the algorithms and strategies. When students lack opportunities to discover connections and relations in mathematics, their creativity and independent thinking can be limited.

Teachers are supposed to be effective in this educational style which according to Minor, Onwuegbuzie, Witcher and James (2002, p. 117) means that they are “subject specialists who are able to select, organize, and deliver content; are efficient and effective in the use of instructional time; and are able to vary their teaching strategies according to student needs. Effective teachers are creative, encourage active student participation, make relevant assignments, arrange for plenty of successful engaged time, are skillful in using questions, promote critical and creative thinking, and use wait time when seeking student response. In addition, they provide feedback, monitor programs and student progress, use both traditional and alternative assessment, and are fair in assessment and grading procedures.”

The teachers use a variety of textbooks approved by the Czech Ministry of Education. Usually, a lecture accompanied by clarifying questions and answers is reinforced with homework assignments. Teachers assign the homework choosing an exercise from the textbook, make worksheets, or design projects with deadlines.

**Methodology**

**Participants**

Originally six teachers of mathematics at elementary level (first through fifth grade where pupils are 6-12 years of age), were identified for the method they
used while teaching in 2017. Two participants could not currently continue and were not included in this study. The other two respondents were added in 2019. Two respondents were intentionally chosen for each approach (A, B for Scheme oriented approach, C, D for Montessori method and E, F for Undesignated method). The teachers work in public schools in Prague (A, B, C, D), Nepomuk (E), and Hostivice (F), Czech Republic. Six teachers of different ages and length of experience (6-20 years) participated in the study. Teachers were asked to an interview for half an hour.

**Data Collection**

A semi-structured interview (Fylan, 2005) with a paper based interview guide was used in this study. The guide included a list of 10 open-ended or closed-ended questions without any particular order. All participants were informed about the background of the study before the interviews started. Two pilot interviews with other elementary teachers were carried out in the first phase. Questions were focused on using assigning homework from mathematics (for instance, “Do you assign homework from mathematics every day?”).

Interviews took place in February 2017 and additionally in February 2019 and lasted from 12 to 42 minutes (30 minutes on average). All interviews took place in a relaxed atmosphere, in a silent room with a minimum of external influences; the interviews were recorded with the approval of the teachers on the voice recorder or videocamera. Transcribed audio or video recordings of interviews were analyzed.

**Data analysis**

Data from semi-structured interviews have been recorded, transcribed and analyzed. Observed phenomena have been categorized, named and described following the principles of Grounded theory (Glaser and Strauss, 1997). Recordings were transcribed verbatim and have been downloaded to the Atlas.ti. Each of the answers was labeled (A1, A2). All the statements were translated by the author of this study.

**Findings**

A number of studies have looked into the topic of homework. This study examined if teachers following any of the three selected teaching approaches assign homework. To answer the first of the research questions in this study, the teachers (A, E, F) admit that they assign homework and demand the same tasks from all pupils.

A2 – All of them have the same homework. We check the homework every working day and I note if somebody does not have it. They need to finish it anyways. The exception is when the children did not
understand it. Parents asked for more homework when the children were in the third grade.

E16 – We have the textbooks for verifying knowledge and for homework.

E26 – The homework is usually an exercise from textbook/workbook.

One of the teachers (F) assigns homework with the purpose of teaching children to study text and to inform parents regarding their child’s performance in school mathematics.

F10 – The children had homework weekly. I prepared it myself. We did it for the parents to see what to study and practice with the children. The weekly homework shows the outline of what they should learn.

F9 – The children are occasionally supposed to study a topic from the textbook, because they should learn to study a text and will need it later with the other teachers.

On the other hand teacher (B) leaves a lot of responsibility on pupils and the homework is more or less just finishing the classroom work. He said that the children can be excused from homework assignments when their work on the school assignments indicates they have mastered the homework material. This is illustrated in the following example.

B5 – I will look at the workbook and record it in the system. If the work from the lesson is correct, they can play soccer in the afternoon. Sometimes they bring corrected work the very next day, sometimes it is flashing in the system till June and it is up to them if they leave it as it is. If they have a lot of unfinished work I just try to force them but I do my best to leave the responsibility up to them. I don’t want them to think that the man wants something. I want them to feel that they want to correct their own work.

The respondents (B, C, D) do not assign homework in traditional way.

B7 – I do not assign any homework. The homework is part of the system where we go back to their mistakes and we learn from them. About five children, and that is what I like, have learned that they correct their work right away. I think that is good. Unfortunately there are also about five children who never do it. The parents rely on the teacher. We need to communicate it more. I want them to understand that they are in charge. On the other hand, I know the reality. One parent comes home at 7 o’clock, the other one at 9 o’clock. It is the question of priority then.

D18 – I do not have homework as it is, meaning that I would assign something extra above what children did at school. I use homework in two cases. It is when I need children to prepare or bring something for a school activity. It can also be unfinished work from the school day. Children have
independent work when they need to finish an assignment from a previous presentation. If the child decides not to work during his independent work time, he needs to take the work home. I note if they bring it back to school but cannot force anybody.

C49 – The children do not get homework. It is because the wooden manipulative material is core and we do not lend it to take home. The homework is to practice something they should already know when they go on vacation.

C51 – They have some assignments like counting the steps to reach the waste separation dustbin. The children should be at school when they are at school and should be at home when they are at home.

**Conclusion and discussion**

Assigning homework every day is familiar to many Czech teachers but effectiveness and the need of it has recently been a popular question in the Czech Republic. The choice of assigning homework nor not can be determined by the directors of school, the pressure from the parents, or the method of teaching.

The respondents in this study, teaching with an undesignated mainstream approach, assign some homework daily. It is consistent to the study of Costa et al. (2016), where respondent teachers, in their majority, assign homework tasks every day and only one of them states that he sets homework only twice a week. The fact that teachers set homework almost every day reveals that even assigning, collecting, and grading homework places high demands on the instructor’s time (Fynewever, 2008). It is still a common practice in this approach.

The teachers, who use the Scheme oriented approach, differ in their opinions about homework. One of them complained about the time for giving feedback and called for an assistant. Sometimes a teacher has a two-day turnaround for getting work back to students. Light (1990) has shown that more immediate feedback (for example, through peer grading) could be more effective for student learning.

The teacher using the Montessori approach sets homework just when the children go on vacation and miss school on purpose. According to Costa et al. (2016) the homework tasks can be different due to the fact that teachers find that some pupils have learning difficulties, it is necessary to reinforce learning, or because there are pupils with special educational needs.

Otherwise, homework is not assigned by Montessori teachers or it is set as practical activities. Sullivan (2007) compared characteristics of early elementary homework for Montessori and traditional schools. The study found that Montessori children were permitted to choose topics of essays and other homework twice as often on average as children in traditional schools (Sullivan, 2007).
The six respondents in this study have been influenced by the teaching method they practice when deciding on homework. They are convinced about why assigning homework in the form they use is, or is not, beneficial. If homework is necessary and appropriate to address learning differences in the classroom, this should be subject to teachers’ thinking.

References


STUDENT THINKING: AN EXAMINATION INTO THE RELATIONSHIP BETWEEN OBSERVING AND TEACHING FIELD EXPERIENCES

Megan Burton

Abstract

This study examines the evaluation of student thinking by teacher candidates when they are observing and when they are engaged in the practice of teaching during a mediated field experience. Teacher candidates co-led elementary science, technology, engineering, and mathematics summer experiences for three weeks. Data from three teacher candidates in one classroom of eighteen children ages seven to eight were examined. Teacher candidates observed students when peers were teaching and also examined work and took anecdotal notes during their own teaching experiences. These data were analyzed using Interpretative Phenomenological Analysis. Findings provide insight into teacher candidates’ perceptions of elementary students’ mathematical thinking when in various field experience roles. This insight can help teacher educators plan experiences that deepen teacher candidate understandings of student thinking.

Keywords: STEM, field experience, teacher education, elementary

Introduction

Effective elementary mathematics instruction involves listening to students and focussing on their mathematical thinking (Aguirre et al., 2013; Bass, 2011). Teachers need to engage, effectively, students in problem solving, rich discourse, and authentic instruction, in order to gain greater insight into the mathematical thinking of their students (Franke, Kazemi and Battey, 2007). However, preparing teachers for this type of instruction, which centres around student thinking can be challenging, because it is rare to see this in the field, and they may have years of experiences as students which opposes this perspective for teaching mathematics. (LeCornu and Ewing, 2008; Shoffiier, 2008). For this reason, it is essential that teacher candidates have experiences that connect coursework and field experiences in meaningful ways to help shift their thinking about mathematics teaching and learning. They need experiences observing student thinking in order to reach, better, students in their future classrooms (Weld and French, 2001). This

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study explores how teacher candidates identify, evaluate, and plan, to respond to student thinking when observing, and when teaching during mediated field experiences.

**Theoretical Framework**

“Teaching is not merely a cognitive or technical procedure, but a complex personal, social, and often elusive set of embedded processes and practices that concern the whole person” (Olsen, 2008, p 5). Teacher’s content knowledge has a direct impact on the instruction they provide to their students (DePiper, Frank, Griffin and Youyoung, 2014). Epson and Junk (2004) suggest that sometimes lack of knowledge about content or student thinking can hinder teachers from acting on their beliefs about teaching and learning. What teachers notice in the classroom setting influences what they are able to utilize as learning opportunities (Sherin, Jacobs and Philipp, 2011). Evidence suggests that novice teachers tend to focus on themselves, whereas more experienced teachers tend to attend to students (Epson and Junk, 2004).

Professional noticing is the “in-the-moment decision making that is foundational to the complex view of teaching” (Jacobs, Lamb and Phillipp, 2010, p. 16). It involves attending to student strategies, analyzing their thinking, and making decisions about responding to student thinking. Understanding student thinking and learning how to utilize this in effective instruction is important in effective teaching and learning (NCTM, 2000). Professional noticing can be learned over time (Jacobs, Lamb and Phillipp, 2010). Supporting teacher candidates in developing mathematical professional noticing skills involves identifying and making explicit productive mathematics learning practices both in coursework and fieldwork (Amador et al., 2016; Cohen and Ball, 2001). This study is grounded on the belief that teacher candidates need early inquiry-based field experiences with positive diverse student populations, where they focus on students as mathematics learners, take teaching risks, and reflect upon the practice of teaching to meet the needs of all learners (Aguirre et al., 2013).

**Methodology**

**Context.** This study examines how teacher candidates (TCs) view student thinking when they are observing in initial teaching field experiences and when they are engaged in the practice of teaching during this same initial field experience. This was the first teaching experience for TCs. They co-led science, technology, engineering, and mathematics (STEM) summer experiences for intermediate elementary students for three weeks. The camp services approximately 150 students from very diverse schools, ethnic backgrounds, and socio-economic status. Over 60% were identified as having financial need. During each four hour day of instruction, mathematics was integrated with the other content areas. In the afternoons, TCs debriefed about the experiences of the day, to analyse student work, and to plan for future instruction.
**Data.** Data from three TCs in one classroom of eighteen children ages seven to eight were examined. TCs involved in this study were Zoe, Joan, and Maria. All three were 21 year old females. They were assigned one grade level classroom to work with for all three weeks, but their roles in the classroom changed each week. TCs planned for five days, taught for five days, observed for five days. There were five days of data collection on student thinking during teaching, and five days of data collection based on observations. Teachers were provided with a template for planning, reflecting, taking notes, and observing students during peer teaching time. These were the data examined, in addition to a semi-structured interview, with each candidate at the end of their teaching experience.

Lesson plans were collected for the entire week they taught and each day they reflected on these lessons and student data. They also made anecdotal notes during the week they taught and reflected at the end of the week they taught. During their observation week, TCs completed a student observation form during lessons, examined work, and made anecdotal notes when peers were teaching. In addition, there was a reflection at the end of the week that they observed as well.

**Analysis.** These data were analysed using Interpretative Phenomenological Analysis (IPA). IPA explores how participants make sense of their world and the experiences they are having (Pietkiewicz and Smith, 2012). Analysis by the researcher of personal accounts given by TCs through reflections, anecdotal notes, interviews, and observations, were paired with the researcher's accounts of the same experiences, which provides double hermeneutics. The process is dynamic as the researcher attempts to gain an insider's perspective into the views of TCs (Smith and Osborn, 2008).

Critical questions were a part of the guidance for teacher candidate observations of peers, observations of students, and reflections on their own experiences. At times additional questions were added to one of the data sets, because of the dynamic nature of IPA, where the researcher is trying to gain an insider perspective.

Initially all data were read multiple times by the individual. Extensive notes on comments, language use, and context were made in each of these readings. Then the notes in phase one were analysed for emergent themes that were still grounded in specific experiences or specific participants. Then connections between emergent themes were identified and grouped into clusters. Finally a narrative account of the interpretations of student thinking was written which highlighted examples of themes within of the identified clusters.

**Findings**

When analysing data from the three participants as individual data sets, several clusters emerged. It was evident that most participants found one area of teaching and, or, learning mathematics, to be central in their reflections, observations, and
lesson plans. Although each was very different, the view of mathematics and mathematical identity of each participant impacted perceptions of student thinking, teaching and learning mathematics.

**Zoe.** Zoe identified student thinking and strategies, but initially focussed on whether responses were “right” or not, and wanted to explain to the student how to correct it, rather than asking questions and providing scaffolding to support student thinking and growth. During her time in the field experience, data reflected growth in this area. She taught the first week and during that time, she tended to lecture and allow very little space for student discourse, unless they were asking for teacher help. However, when analysing student work and observing them the following week, she was able to identify the need to allow students to discuss strategies and experience ownership in their learning. Her reflections indicated growth in perceptions about teaching and learning, but she was not provided an additional place to practice this change in thinking during the semester.

After the first day of teaching, Zoe was watched and reflected on a brief video-clip from the day’s instruction. During this video clip, she told a student an answer. She explained, “It is difficult to know when to help students and how to help them.” She communicated that mathematics should be easy and fun. As a student, she had not liked mathematics and thought she was bad at it, therefore she wanted to avoid that with her own students.

When observing students’ learning during the field experience, she noted times that teachers either left a student frustrated or told the student what to do. One time she asked a student to apply the solution, that they had been told, to a new concept and the student could not. This allowed Zoe to see how telling limited a student’s ownership in the learning process. This reflection correlates with Abell (2006) who noted that teacher candidates need experiences with strategies and pedagogy learned in coursework in order to more fully visualize its place in authentic classrooms.

**Joan.** Joan’s reflections focussed on management and behaviour. It was very difficult to draw her attention to the mathematics. For example, when she observed students struggling to count money in a lesson that involved regrouping, her observation notes reflected on student misbehaviour and noted they were off task and “not trying,” rather than exploring if the content could be the instigator in these behavioural issues. She struggled to identify student thinking, but focussed on student actions. She also failed to identify what students did know, which could have helped in her planning for the following week. She expressed challenges with multi-tasking and keeping students focussed.

Before her teaching experience, Joan was concerned that the students wouldn’t “like or listen” to her. She was excited to help a student who was frustrated with
a geometric activity that involved robots. The student was upset and off task. Joan went and asked if she could help and he told her how frustrated he was. Joan was able to help him through each step of the task. However, when discussing with her peers, she admitted that she told rather than scaffolded in this example. She felt that the emotions and frustrations of the student justified her giving the information.

**Maria.** Maria loved science and saw mathematics as a tool to explore the science. However, she struggled to identify the student thinking and learning in relation to mathematics. For example, she planned on students using coins to buy materials for a science experiment. When she noticed that many students struggled with issues involving regrouping, she provided a calculator to help them, rather than adjusting plans and exploring this mathematical content. When asked about this opportunity, she said that she felt they would lose interest and wouldn’t get to the science experiment if she taught the mathematics. This demonstrates the importance that she put on the science over the mathematics.

When observing Maria noted the importance of knowing mathematical content in order to focus on the science. She struggled in identifying ways to make the mathematical content accessible to the students that were confused. This relates to the mathematics needed for teaching elementary students (Bray, 2011; Fisher et al., 2014). Maria stated:

> I have always thought mathematics was easy. I never expected them to have the problems they have. I thought if I just showed them they would get it and we could move on to science, because it is fun. Now I realize it isn’t the same for everyone. I need to really pay attention and learn from others and find ways to help this make sense to them.

During her time observing students, Maria began to observe that students could have some understanding of content and still get the answer wrong. This was a powerful moment in her professional development and her views about teaching mathematics. Maria commented on observing:

> When I first heard the term productive struggle it didn’t make any sense to me. I thought the word “struggle” in itself is a teacher’s role to minimize. Also, I thought you either understood or you didn’t, all or nothing. I now realize that isn’t the case. When given the chance to work through it on their own, they realized how much they knew and what they could do, but the teacher watched from afar to ensure they weren’t left without tools to solve it.

**Discussion**

This analysis of data from three teachers during an initial field experience provides insight into their struggles in professional noticing. However, all demonstrated growth and an awareness in their final reflections, that students are all different and that teachers need to be aware of where students are
mathematically, in order to best support their mathematical growth. All were able to benefit from their peers, the feedback they received, and the experiences in the early inquiry-based field experience with positive diverse populations. They were able to take risks and focus on students as mathematics learners, while also reflecting upon the practice of teaching to meet the needs of all learners (Aguirre et al., 2013). Findings provide insight that supports the benefit of these experiences, but also illuminates the challenges that TCs face in their initial understanding of student thinking and the productive struggle that occurs during learning. It also shows some of the challenges TCs face when examining mathematics that is integrated into other subject areas. There were changes in teacher candidates' perceptions of elementary students’ mathematical thinking throughout their time in the various roles during the field experience. This insight can help teacher educators in planning experiences that deepen teacher candidate understandings of student thinking.

References


Problem-Posing Strategies: Showcasing Elementary Student Responses

Julia E. Calabrese, Ashley M. Williams and Mary Margaret Capraro

Abstract

Problem-posing instruction is becoming an important component in mathematics classrooms. The aim of the present study was to demonstrate how research-based problem-posing strategies can be enacted in elementary mathematics classrooms. Participants (n=137) were second-, third-, and fourth-grade students. Four research-based problem-posing strategies: manipulatives, games, real-life applications, and
student interest were implemented. Researchers qualitatively analyzed student responses. Overall, the four strategies used encouraged students to create reasonable and solvable word problems

**Keywords:** elementary, instructional strategies, mathematics education, problem posing

Problem posing is an instructional strategy in which students can use their creativity to write their own word problems or modify previously existing problems (Silver, 1994). A growing number of educational researchers have advocated the use of problem-posing instruction in mathematics classrooms; in fact, it is currently included in the common core standards (National Governors Association Center for Best Practices, 2010). However, before choosing whether to utilize this instructional strategy, teachers should become familiar with strategies for effectively incorporating problem posing into their classrooms.

**Strategies for Implementing Problem Posing**

When attempting to incorporate problem-posing instruction into existing lessons, teachers may need assistance on how to facilitate a problem-posing activity that involves more than simply asking students to write a problem. Though these are not all inclusive, some research-based pedagogical strategies for problem-posing instruction include the use of manipulatives and games and the incorporation of real-life applications and aspects of student interest. These strategies when incorporated with problem posing can help students use their interests to create their own creative problems.

**Manipulatives.** The use of manipulatives in mathematics can help students develop a more meaningful and concrete understanding of mathematics content (Clements, 1999). Manipulatives offer a tangible perspective when concepts may seem too abstract (Moch, 2002). Therefore, it makes sense that using manipulatives throughout the problem-posing process could allow students to better visualize a problem scenario (Chapman, 2012; English, 1998; Munroe, 2016; Rosli, Goldsby and Capraro, 2015). In fact, researchers (Ulfah, Prabawanto and Jupri, 2017) have even stated that students may potentially pose better problems if they have access to manipulatives during the writing process.

**Games.** Incorporating games in the mathematics classroom can add an element of fun while still fostering an effective learning environment (Afari, Aldridge, Fraser and Khine, 2013; Booker, 2000; Plass et al., 2013). What teachers may not immediately realize is that games can be used during problem-posing instruction (Kalmpourtzis, 2019). Using games can help foster interest students with problem posing (Chang, Wu, Weng and Sung, 2012). Increasing student interest in mathematical problem posing is important and certain games can make this goal more achievable.

**Real-life Applications and Student Interests.** When students can see that mathematics is relevant to real-life applications (Afari et al., 2013) and to their
personal interests (Winograd, 1991), they tend to enjoy the material more (Walkington and Bernacki, 2015). The National Council of Teachers of Mathematics (NCTM, 2000) has stressed that teachers should teach in such a way that students can see how mathematics exists within their everyday lives. Through the use of problem-posing instruction, students have the opportunity to choose what topics are the subject of the problem, allowing them to incorporate their own interests (Silver, 1994). Using more personalized word problems has the potential of motivating students to learn mathematics (Walkington and Bernacki, 2015). Problem-posing instruction provides an outlet for students to apply and engage with their interests in an educational mathematics context and can be used to help students realize that math exists everywhere.

Teachers may introduce their own techniques for implementing problem-posing instruction into their classrooms as they become familiar with the instructional approach. Problem-posing instruction has many benefits, including improving student interest in mathematics (Walkington and Bernacki, 2015) and increasing students’ mathematics achievement (Cankoy, 2014; Chang et al., 2011; Sugito, Susilowati, Hartono and Supartono, 2017; Sung, Hwang and Chang, 2016). Therefore, it is important that more educators become aware of the types of problems that can be posed and how to smoothly introduce the approach within their classrooms. As problem-posing instruction becomes more widely implemented in classrooms, it is likely that a greater variety of problem-posing approaches and benefits will be realized. The purpose of the present study was to model effective strategies of implementing problem-posing instruction in the classroom.

**Method**

Rather than one singular textbook strategy, problem-posing instruction can be implemented in many ways. An instructor’s choice for problem-posing instructional strategies may depend on appropriateness for the class, the lesson, or his or her personal preference. The aim of the present study was to demonstrate how research-based problem-posing strategies can be enacted in elementary mathematics classrooms.

**Participants**

This particular study took place in two elementary schools located in the southwestern United States. Participants were second-, third-, and fourth-grade students during the 2017-2018 ($n = 64$) and 2018-2019 ($n = 73$) school years. In total, this portion of the study contained 137 students in the problem-posing intervention. As reported by the Texas Education Agency (2018), the combined demographics of the four elementary schools in the participating in the larger entire study during the 2017-2018 school year included 15.5% African American students, 45.9% Hispanic students, 35.2% White students, and 3.5% students who identified as other.
The study took place over the course of two years. Each year, two elementary schools participated in the problem-posing intervention. Students remained in intact classrooms as designated by the school district. Each of the chosen classrooms contained both a regular classroom instructor as well as a preservice teacher. The regular classroom teacher was responsible for day-to-day instruction, covering district scope and sequence topics within the state standards. The preservice teachers in each of the classrooms implemented the problem-posing intervention.

**Intervention**

For every semester that the study took place, the researchers designed lessons based on existing research on problem-posing strategies. The lessons slightly differed by grade appropriateness based on grade level scope and sequence. All lessons were designed and aligned with grade-level state standards as well as the school district’s overall scope and sequence. A lesson consisted of a review portion followed by a problem-posing intervention. During the review portion, the preservice teacher led the students in a brief activity and example problem. During the problem-posing intervention, the students were asked to write their own problems based on the given concept. Each week, the preservice teachers received training at the university from the researchers on lesson administration to increase consistency and fidelity of implementation among the students. Lesson plans and materials were created and distributed by the researchers. Preservice teachers collected all student responses and returned them to the researchers. In total, 10 lessons were conducted in each classroom. For the purposes of this study, one lesson was selected to model each of the four research-based strategies for implementing problem posing in mathematics classrooms: 1) manipulatives, 2) games, 3) real-life applications, and 4) popular cultural references that would be of student interest.

**Analysis**

Researchers qualitatively analyzed student responses. Responses could be analyzed and classified as either correct or incorrect. For the purposes of this study, the researchers chose to highlight selected responses based on their uniqueness and reflectiveness of the problem-posing intervention.

**Results**

An analysis was completed for all three of the grade levels. Researchers limited the analysis to four of the ten lessons. This was to align with each of the previously mentioned research-based problem-posing strategies: manipulatives, games, real-life applications, and student interest. Student responses were transcribed and are included in Table 1.
Manipulatives (Erasers)

2nd Grade
I had 12 paints. [Student] gave me 11 more paints. How many do I have now?

3rd Grade
I baked 23 cupcakes. I bake some more. Now I have 92. How many cupcakes did I bake to get 92?

4th Grade
The treasure box has 1 cat eraser, 3 witch hats, and 2 pumpkins. If [Student] earned all the witch hats, what fraction of the erasers did she earn?

Games (Cards)

2nd Grade
I have 97 candy. [Student] gave me 3 more. How many candy do I have now?

3rd Grade
Mrs. [Teacher A] and Ms. [Teacher B] are playing cards. [Teacher A] has 334 cards and [Teacher B] has 298 card then Teacher A gives her 52 more. How many does Teacher A have now?

4th Grade
Jack spent $75.7 on balls, and then he spent $7.2 on candy. How much money did he spend?

Real-Life Application (Grocery)

2nd Grade
I have string cheese. It cost 30¢. I have small lemons. They are 33¢. How many is the ¢ in all? The butter cost 13¢. The coconut water cost 36¢. How many is it in all?

3rd Grade
Macey needed to buy juice and fruit for breakfast this week. The apples were 23¢, the oranges were $2.98, the grapefruits were 71¢, and the orange juice jugs were $3.19. How much money does it cost in all?

4th Grade
The bread usually cost $3.20 and now $2.68. The BBQ sauce costs $1.50 and now $1.00. Which item can you save more money on?

Student Interest (Zootopia)

2nd Grade
Judy Hops is 3 inches tall. Mr. Fox is 5 inches tall. How much taller is Mr. Fox than Judy Hops?

3rd Grade
How tall is Janine and Pennington combined?

4th Grade
What is the best estimate of the difference between the elephant’s height and the rabbit’s height?

Table 1: Selected Student Responses
Manipulatives. The use of manipulatives in mathematics classrooms is a common practice that involves providing students concrete objects to manipulate, which assists them in visualizing and posing problems. Taking advantage of the known benefits of manipulatives, researchers developed lessons using Halloween erasers. The second and third grade students used the erasers and a part-part-whole board to review addition and subtraction while the fourth-grade students reviewed and created fractional parts using the erasers. After they reviewed the concepts, students were then asked to pose one- or two-word problems. The second- and third-grade students were asked to pose addition or subtraction problems, and the fourth-grade students were asked to write a problem involving fractions.

All three students, whose responses are represented in Table 1, wrote realistic, solvable problems involving tangible items. The second-grade student wrote a two-digit addition problem with the total unknown, while the third-grade student’s two-digit addition problem was slightly more advanced, leaving the change unknown. Both problems involved tangible items and addition involving change. The fourth-grade student used the provided manipulative (Halloween erasers) to write a problem involving fractional parts.

Games. The use of games in the classroom can add an element of fun and excitement, engaging students in the activity or lesson. With the hope of engaging students in a problem-posing lesson, researchers developed a lesson in which students used playing cards to build addition and subtraction problems. The second-grade students built two-digit by one-digit expressions, the third-grade students built three-digit by two-digit expressions, and the fourth-grade students created expressions with one decimal place. Using the expression created, they were then asked to pose one- or two-word problems.

The selected cases were all reasonable scenarios with solvable problems. The second-grade student wrote an addition problem about candy with an unknown total. The third-grade student wrote a subtraction problem that required some critical thinking skills because solving the problem would require one to carefully read the question and notice that one must subtract the amount of cards given away from Teacher A to answer the question rather than adding to Teacher B’s cards to find out her total, which is a common question asked. The fourth-grade student wrote an addition finance problem about purchasing items and finding a total cost.

Real-life Application. When students use mathematics in relevant real-life applications, they begin to see a purpose for doing the mathematics. Through the use of problem-posing instruction within a grocery store setting, students chose grocery items that they were interested in purchasing. In this lesson, second-, third-, and fourth-grade students first reviewed pennies, nickels, dimes, and quarters and worked an example problem together. They were then asked to pick cards with items and prices from grocery ads. Next, students were asked to pose
their own problems based on the objects they choose. Lastly, they wrote their problems on their sheet.

As shown in Table 1, the second-grade student merely looked at the advertised grocery items and picked four unrelated items (string cheese, lemons, butter, & water) and was able to use them in two different problems asking two similar questions, “How many is the cents in all?” These two problems were solvable.

The third-grade student wrote his grocery problem around a theme: food for breakfast for the week focusing on fruits and juices. Again, this student chose four grocery items but wrote one problem stem requiring one question to make it a solvable problem. It was interesting to see how the student set up the problem to solve. He added the dollars ($2 & $3), then the cents (98, 71, & 23 cents), and then added both together to obtain the correct overall amount. This student in the problem set up listed the dollars under the dollar sign and those with cents under the cents sign. When he added the amounts, he wrote the answer without dollar or cents signs, but at the bottom of the problem, he was able to add the two amounts and in the final answer did include the dollar sign.

The fourth-grade student was able to write an interesting and unique problem comparing the prices of two items from the grocery advertisements (BBQ sauce & bread) at regular cost and on sale. He then asked, “Which item can you save more money on?” - the bread, which usually cost $3.20 and is now $2.68, or the BBQ sauce, which usually costs $1.50 and is now $1.00.

**Student Interest.** To appeal to student interest, the researchers incorporated characters from the popular movie *Zootopia* into a lesson on graphing. Each of the characters was shown on a graph measuring their heights. Students were guided to write a problem using the graph. There were no further guidelines or restrictions in the writing.

After this lesson, the preservice teachers reported how excited the students were to see the graph. In two of the responses, the students used the names of the characters in their problems even though the character names were not provided in the image. Because the names were not required information, this shows the students were able to incorporate their own outside knowledge into the lesson. Additionally, one of the students went as far as to write and solve 10 questions even though he or she had only been asked to write one or two problems.

**Conclusion**

The implementation of strategies such as using manipulatives, games, real-life applications, and students’ interest in problem-posing activities resulted in successful student creation of word problems. Use of these strategies in the current study demonstrated how teachers can implement problem-posing instruction in elementary mathematics classrooms. From examining the student-created
problems, researchers were able to identify positive aspects of the activities as well as topics for future research.

**Manipulatives.** During the Halloween eraser lesson, students were given manipulatives to use when forming problems, which influenced their responses in that their problems were realistic using concrete objects. It is possible the use of manipulatives helped the students to visualize the scenario, influencing the solvability of the problems.

**Games.** The playing cards lesson provided students game cards to build expressions, adding a fun factor to the activity. When selecting a fourth-grade response, the researchers were able to notice that many of the word problems posed did not use decimals in a logical way (i.e., 39.7 markers or 7.2 video games). One added benefit of PP instruction is that it can allow teachers to gain insight into students’ learning gaps/misconceptions related to the content and concepts they are attempting to apply in their posed problems. The selected response demonstrated that this particular student did understand decimals and used the mathematical expression to write a finance addition problem with an unknown total.

**Real-life Application.** The grocery store lesson included relevant real-life applications to engage students in the problem-posing activity. Students were excited to find items in the newspaper advertisements their instructors provided to them that actually contained foods or products they ate or used in each of their household families. This led them to develop meaningful, solvable problems as they were engaged in the grocery lesson. This can be seen in the problem posed by the fourth-grade student, which included the added element of being “on sale” and connected the current price to the sale price. This student added dimension to the posed problem (not suggested by the instructor), demonstrating that real-world situations can stimulate both interest and creativity in mathematics.

**Student Interest.** In the Zootopia lesson, the students’ enthusiasm toward this exercise shows that incorporating students’ interests into mathematics lessons is beneficial. One of the students even went above and beyond the required task. Making a lesson relevant and interesting for students is a common practice and continues to have promising results. This strategy encourages student involvement, excitement, and engagement in mathematics.

All four of the strategies used encouraged students to create reasonable and solvable word problems, as was evidenced through the examples provided in this study. In one area, decimals, it was noted that students had difficulties creating real world problems that were reasonable. Implementing problem posing allowed teachers to pinpoint these specific issues with decimals. In the future, teachers can try to identify and correct specific misconceptions quickly by discussing real world use of decimals before asking students to pose decimal problems. For future research, to better understand the influence of these strategies on the students’
problem posing, further insight is needed; informal interviews with students, directly after they have completed their work could be an excellent way to gain additional insights into student thinking.

References


**ADDRESSING THE ISSUE OF TRUST IN ELEMENTARY TEACHERS’ MATHS-SPECIFIC EDUCATION: ANFOMAM PROJECT**

*Valentina Celi, José Ignacio Cogolludo, Elena Gil Clemente, Inmaculada Lizasoain*, Ana Gasca Millán, José Antonio Moler and Luig Regoliosi

**Abstract**

To improve primary school teachers' maths-specific education at university, our project will develop a series of workshops, as ready-to-use instruments, which closely consider children's way of learning and their relationship with mathematics. Thus, the interest of participants in children is exploited in sessions which take into account both their

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professional work as teachers and their own childhood experiences. The aim is to help participants to evolve in the key aspect of trust. The paper describes the objectives and first results of the ANFoMAM project, supported by the Erasmus Plus Programme in the area of strategic partnerships for innovation in higher education in Europe.

**Keywords:** university mathematics education, pre-service mathematics education of primary teachers, in-service professional development of primary mathematics teachers, Trisomy 21, education of children with intellectual disabilities

**Introduction**

The ANFoMAM project takes its name from the Spanish version of its title, “Learning from children to improve Maths-specific primary teachers’ education”. The idea of ANFoMAM arose from the Symposium *Numeracy and Beyond* in the 5th International Congress of educational sciences and development (Santander and May 2017), when we interchanged different experiences stemming from work carried out to improve the training of both pre-service and in-service primary school teachers in mathematics in France, Italy, Norway and Spain (Celi and De Simone, 2018; Campión Arrastia et al, 2017; Lekaus et al., 2015).

In these countries, trainee teachers report a poor experience of maths courses in primary and secondary school that often includes feelings such as annoyance or distress together with a vision of mathematics reduced to pure calculation and mechanical procedures (Gil Clemente and Millán Gasca, 2016).

Avoiding the false debate between improving teachers’ mathematical training or focusing on teaching didactical techniques, all the initiatives presented at the Congress in Santander pursued the common goal of providing teachers with both a deeper understanding of the basic concepts of mathematics and a knowledge of effective teaching resources for the classroom (Rico, Gómez and Cañadas, 2010).

As teachers in charge of elementary mathematics in further education, our common view was that it must be the university that provides these trainee teachers with new mathematical learning and teaching experiences, so that they can transmit to children confidence and enthusiasm to learn. But, what kind of course would be able to provide such experience to trainee teachers?

We agreed that a workshop format would be suitable for this purpose because it encourages a more active involvement of the participants in the activities than the ordinary classes. Consequently, the main goal of our project is to design didactical material for use in the above-mentioned format, to be offered both as complementary activities to standard university courses in mathematics and for updating in-service teachers (Catalán et. al, 2019).

The workshops will be inspired by the training interventions shown in Santander, which deal with different aspects of the learning of mathematics, all focussing on creating an adequate pedagogical environment. Much inspiration was drawn from the mathematical activities with children with Trisomy 21 designed in the
Sesdown association in Zaragoza (Spain) (Gil Clemente, 2016). These activities do not require any previously-acquired cognitive skills from children; on the contrary, mathematical activities are applied in order to develop children’s abstract thinking and imagination, and to enhance their self-fulfillment and their relation with the world around them. We think that the way in which these children are learning maths can shed light on the way of learning maths of any child in general.

During the project, the mentioned workshops with Down Syndrome children will be recorded in videos, which will be part of the didactical material addressed to teachers at formation. In this way, teacher trainees will have the opportunity of watching a specific way of learning, in order to learn about maths learning in general. In this way, the diversity will be seen as an opportunity for growth for all the students.

The institutions associated with the project, which has been co-founded by the Erasmus + program of Strategic Partnership in the field of higher education, are: Université de Bordeaux (École supérieure du professeurat et de l'éducation d'Aquitaine), France; Università Roma Tre (Department of Education), Italy; Universidad de Zaragoza, Universidad Pública de Navarra (Spain); Sesdown Association, which investigates on education of children with Trisomy 21 in Spain and Tokalon Association, working in the area of professional development of in-service teachers, in Italy. The coordinator is the Public University of Navarre in Pamplona, Spain.

**ANFoMAM project**

**Goal of the project**

Often, primary teachers have a bad relation with mathematics because of their own experience during childhood and secondary school (Margolinas, 2007). The goal of the project is, firstly, to know the beliefs and attitudes that both pre-service and in-service teachers have in relation to mathematics by means of six questionnaires, each of them designed for a specific aspect of the discipline: (i) Understanding of arithmetic algorithms; (ii) solving arithmetic problems; (iii) relationship between arithmetic and geometry; (iv) reasoned computation and use of the calculator; (v) history of mathematics and its teaching and (vi) geometric constructions and solving geometrical problems.

Then, we intend to design the didactical material necessary to implement six workshops (or *ateliers*) for training in each of the specific areas to both prospective and to in-service teachers.

Each of these workshops will be composed of several segments or sessions with a total duration of between ten and fifteen hours. In the case of pre-service teachers, the workshops will serve as a complement to standard university courses in mathematics. In addition, the six workshops will constitute a whole updating
course for in-service teachers. During the project, of three years’ duration, the workshops will be implemented in a pilot phase with some regular groups of students of the associated institutions. So, the characteristics of the groups will be different at each of them, depending on its own organization.

Although each workshop will address a different aspect of children’s maths education, all of them will share a same pedagogical frame. This frame, which must be better defined during the project, will take into account the following aspects:

1. A close sight at children’s way of learning, analyzing not only their conceptual or procedural difficulties with maths but also which tasks make sense for them (Donaldson, 1987)
2. Search for a deeper understanding of the basic concepts (objects and relationships) in mathematics together with their roots in human experience (Millán Gasca, 2016)
3. A preference for relational learning before instrumental one (Skemp, 1976)
4. Promote active methodologies that encourage students’ participation (Campión Arrastia et al., 2017)
5. A real consideration of the inclusion of children with special needs in the classroom (Gil Clemente and Millán Gasca, 2016)

Although the workshops are addressed to teachers at (initial or continued) formation, we intend that the designed activities can be easily translated to the school classrooms. In this way, participants will learn maths at the same time that they get practical resources to teach maths to children.

**Methodology**

The methodology of the workshops is inspired by research in the Laboratory of Mathematics for primary school at the Department of Education in Roma Tre University. The workshops will simulate a primary school classroom where adult activities are combined with the design of activities addressed to children. *Mimesis* (as in theatre and performance arts) is awakened as a pedagogical method (Scaramuzzo, 2016), following research in the MimesisLab (Laboratory on the Pedagogy of expression) in the above-mentioned Department of Education: Students will place themselves imaginatively at the children's point of view, which will help them to understand which tasks make human sense for them and which difficulties learning may present for children.

At the same time, students will learn to apply mimesis to the mathematics classroom at school, designing tasks that require children to act in ways which are in line with very basic human purposes and intentions, as suggested by Donaldson (1987). This kind of approach has already been used in the mathematical workshops that Sesdown Association carries out regularly with Trisomy 21 children (Millán Gasca and Colella, 2017) in which children may feel like if they were, for instance, tightrope walkers in a circus, to notice the characteristics of a straight line, or tiger tamers, to distinguish the inside of his hoop from the
hoop itself. These activities take advantage of children's capacity of understanding the whole situation, in order to make conscious thought emerge from their naïve conceptions.

The university students of teacher degrees that have participated as voluntaries in Sesdown workshops report a new perception of maths that are giving them a deeper understanding of the basic concepts (objects and relationships) in mathematics together with their roots in human experience (Gil Clemente, Millán Gasca, 2016). At the same time, the participation in the workshops is providing students with a lot of resources to teach mathematics with pleasure, strength and effectiveness. The project intend to extend this learning opportunity to pre-service and in-service teachers by recording Sesdown workshops in videos that will be included in the didactical material designed in the project. In this way, future teachers will have the opportunity of analysing them in order to acquire didactical resources to work not only with special needs children, but also with every child in an inclusive way.

Besides the analysis of the videos, some specific tasks will be designed for each workshop, sharing the aims described in Section 2.1:

i) **Understanding of arithmetic algorithms.** Activities will be designed to compute the four basic operations with material and graphic support in order to the understanding of algorithms linked to properties of numbers. At the same time, there will be an analysis of the convenience of choosing a particular algorithm and a specific support in each case.

ii) **Solving arithmetic problems.** We will design several mathematical situations that allow students to state arithmetical problems. The emphasis will be put on discovering and representing graphically implicit relationships between magnitudes. Different resolution strategies will be also analysed.

iii) **Relationship between arithmetic and geometry.** Design of situations that show natural connections between these two disciplines, such as repetition of equal elements, presented both in multiplication situations and in measurement problems; comparison of quantities that can be represented by using geometry; or composition and decomposition situations, occurring in measurement tasks.

iv) **Reasoned computation and use of the calculator.** Design of activities that show the mathematical knowledge underlying some calculus strategies together with tasks that improve the combined use of mental calculation and calculator for exploring numerical facts.

v) **History of mathematics and its teaching.** Reflection about the dynamical nature of maths for giving students a new perspective not only on mathematics but also on maths teaching, making them capable of analysing the way they have been taught and of finding grounds and resources to do it in another way in their professional activity.
vi) **Geometric constructions and solving geometrical problems.** Discovering of invariant properties of figures by movements, making geometrical constructions that cause surprise and aesthetical pleasure to students while help them in the building of the abstract geometrical space.

**How to enhance students’ trust**

Donaldson describes the tasks proposed to encourage children to learn a certain discipline as those “the child will be able objectively to do well, but not too easily, not without putting forth some effort, not without difficulties to be mastered, errors to be overcome, creative solutions to be found” (Donaldson 1987, pp. 115). These kinds of activities, together with a sensitive and accurate response of the teacher to children’s errors, will help them to acquire confidence and energy.

Translating these ideas to further education, these are the kind of tasks we intend to propose pre-service and in-service teachers during the workshops in order to give them the necessary trust\(^1\) to teach maths in a pleasant and effective way.

During our experience as teachers at university, we have realized that, on a purely rational level, ordinary classes provide a lot of things that should help students to be teachers, such as a good understanding of basic mathematics or its implications in the maturation process of a child. In addition, the university professor, by means of their listening attitude and optimism, may transmit to students a feeling of confidence based on good arguments, tested experiences, efficacy evidences, etc.

But this is not enough! It is also necessary for both future and current teachers to acquire a feeling of *trust*, different from faith but deeper than confidence, more instinctive than rational, that we expect that workshops can provide. Covey and Merrill (2006), referring to professional success in general, write that “trust is the most overlooked, misunderstood, underutilized asset to enable performance. Its impact, for good or bad, is dramatic and pervasive.”

In the context of education, Orón Semper and Blasco (2018) review the literature existing about the importance of the quality of the relationships between the teacher and the student even at university. Citing Portelli (1993, pp. 345), they say that a collaborative activity teaching requires trust and they demand teachers believe in the student's potential to create a relationship of trust and respect.

We expect that the workshops designed and implemented during the project provide students with new experiences that change their beliefs about the nature of maths and about their own capacity of learning it, making participants reconcile with their past, which is operating at an unconscious level.

For this purpose, it is necessary to design activities that lead to participants leaving the sessions with the idea that they will not fail in performance, or at least, they do not have a fear of failing because they are willing to take the risk of teaching; the result is worth it.
But this will only be possible if there is also an important change in the way that the workshop instructor looks at their students. Instead of focusing only on their knowledge of mathematics, the activities designed for the workshops must let the professor become a witness to their students' interest in children as well as their ability to teach them in an effective way.

Results

At the moment, the team has developed some questionnaires designed to detect some conflicting areas in the training of pre-service teachers, which will be the starting point for developing the six workshops planned.

Besides the workshops-related questionnaires, a general questionnaire about students’ beliefs and attitudes has been given to pre-service and in-service teachers in France, Italy and Spain. An initial analysis of the results shows three different profiles among the students: a first group that shows a strong lack of motivation linked to a static conception of mathematics; a second one with an instrumental vision of the discipline, linked to effort and perseverance; and a motivated group of students with a dynamic vision of mathematics, as a subject that provides enjoyment and personal growth. This last position is more frequent among in-service teachers than among university students. Some differences also appear between countries, but further detailed analysis is necessary to specify what these are.

In a complementary way, the team's interest in the teaching of maths to children with special needs is resulting in the design of new activities capable of bringing out these children’s mathematical competence. In this first phase of the project, the activities pay special attention to basic geometrical concepts and their relationships, developing their performance in simple tasks such as drawing a straight line or a circle or comparing objects.

Implications

The main goal of the project is to make some significant contribution to the improvement of the mathematics training for future teachers, without forgetting the updating of in-service professionals, providing them with practical didactical resources which give them a new way of viewing maths while providing both elementary maths contents and also the point of view of children on these contents. Every workshop virtual box will be uploaded on Erasmus + Project Results Platform in order to be downloaded free; it will include materials and detailed instructions regarding their use with groups of prospective or in-service teachers.

At the moment, we are testing several materials in formation workshops addressed to in-service teachers or in mathematical workshops especially designed for children.
At the end of the project, we envisage making several dissemination events to divulge the results that have been obtained.

Acknowledgement: The paper was supported by the Erasmus+ project 2018-1-ES01-KA203-050986 entitled “Learning from children to improve primary school teachers’ maths-specific education”.

References


**PROSPECTIVE ELEMENTARY TEACHERS’ CONCEPTIONS OF AUTHENTIC ASSESSMENT MATHEMATICS TASKS**

*Olive Chapman*, Kim Koh and Juan Luis Piñeiro

**Abstract**

Mathematics teachers need to hold knowledge of alternative ways of assessment that are aligned with the objectives of reform-oriented mathematics curriculum. This paper reports on a study to explore prospective elementary teachers’ conceptions of authentic assessment and authentic assessment tasks in relation to teaching and learning mathematics as the first stage of a project aimed at helping them to improve their assessment literacy. Findings, based on data consisting of interviews and participants’ creation and analysis of mathematical tasks, indicate limitations in the participants’ conceptions and the need for intervention to help them to clarify, extend, and reconstruct their conceptions of assessment and coherence among assessment, curriculum, teaching, and learning.

**Keywords:** authentic assessment, elementary teachers’ conceptions, tasks

Mathematics curriculum reforms have necessitated careful reconsideration of the ways in which teachers evaluate students’ progress and mathematical understanding. This has resulted in significant attention to formative assessment as a tool for improving the teaching, and for providing better conditions for students’ learning, of mathematics (e.g., Fennell, Kobett and Wray, 2017; Silver and Mills, 2018). However, implementation in the classroom is dependent on the teacher and his or her knowledge and experience with non-traditional assessment practices. Research on prospective mathematics teachers have indicated that, without appropriate intervention, their knowledge and beliefs of assessment

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tended to be more about traditional or summative assessment (e.g., “paper and pencil tests”) and much less about alternative or formative assessment (Coffey, 2000; Santos and Teixeira, 2018; Wallace, 2012). Their perspective of assessment included procedural tasks that were used for grading purposes or to identify topics to review. They were more familiar with the written test, felt insecure with other instruments, and in general, had narrow views of assessment. In some cases, even with exposure to alternative assessment, many still showed superficial changes in their thinking of non-traditional assessment. This suggests the need to continue to explore their conceptions and ways to support them to improve their assessment literacy in ways that are well aligned with the objectives of reform-oriented mathematics curriculum. This paper reports on the first phase of a three-year funded project that investigates an intervention approach to help prospective elementary teachers to understand authentic assessment and develop expertise in the selection, adaptation, and design of mathematics authentic assessment tasks. The focus here is on the prospective elementary teachers’ conceptions of authentic assessment and related tasks prior to the intervention. Specifically, the key research question addressed is: How do the participants understand authentic assessment and authentic assessment tasks in relation to teaching and learning mathematics based on their prior experiences?

**Theoretical Perspectives**

Our focus is on authentic assessment and authentic assessment tasks. Authentic assessment is a form of formative assessment, that is, it yields information that helps to improve teaching and learning (Lajoie, 1995). However, it is unique because it always relates to the real world (Frey and Allen, 2012). In particular, it consists of authentic assessment tasks, that is, tasks that replicate real-world challenges and standards of performance that experts or professionals typically face in the field (Wiggins, 1990). They involve realistic situations, questions or issues that might actually occur in a real-life situation and use of realistic tools and resources (Lesh and Lamon, 1992). As such, the tasks must include contextualized, complex intellectual challenges involving students’ application of knowledge in messy, ill-structured contexts. Thus, in attempting mathematics authentic tasks, students are required to think and act like real mathematicians who are doing mathematics or like professionals who use mathematical concepts and procedures to solve real-world problems. (See Koh, 2011 and Koh and Chapman, 2018, for examples of such tasks in mathematics, which are too long to replicate here.)

In our project, we adopted this perspective of authentic assessment and authentic assessment tasks in which students are required to demonstrate meaningful application of essential mathematics knowledge and skills. Building on this perspective of authentic assessment, we developed an intervention for supporting prospective teachers learning of authentic assessment in mathematics based on the “authentic intellectual quality” framework (Koh, 2011). The Authentic
Intellectual Quality framework consists of eight criteria and their respective elements that can be used to design and/or evaluate authentic assessment mathematics tasks: (1) depth of knowledge, (2) knowledge criticism, (3) knowledge manipulation (e.g., interpretation, analysis, application/problem solving, construction of new knowledge), (4) extended communication, (5) clarity and organization, (6) making connections to the real world beyond the classroom, (7) student control, and (8) explicit performance standards or success criteria. In the phase of the project reported here, we used these Authentic Intellectual Quality criteria to determine the prospective teachers’ initial level of understanding of mathematics authentic assessment and assessment tasks.

Research Methods

A qualitative methodology involving a grounded theory approach (Corbin and Strauss, 1990) to explore the prospective teachers’ conceptions about authentic assessment was used to frame the study. Participants were 16 prospective elementary school teachers (grades 1 to 6) who volunteered to participate in the study. They all are expected to teach mathematics during their field experiences and in their future teaching. They all hold university degrees but not in mathematics or mathematics-related disciplines. They were at the end of the second term of their two-year Bachelor of Education program and had completed a required assessment course in this term. While this course included a focus on formative assessment, it did not address authentic assessment specific to mathematics. They all had taken a required Science, Technology, Engineering, Mathematics (STEM) course that exposed them to authentic tasks and four of them had taken a mathematics education course that did not include assessment as a topic.

The intervention consists of two one-week sessions in May 2018 and May 2019 immediately after the end of term 2 and term 4, respectively. Details of the intervention are not relevant to this paper, which addresses the participants’ pre-intervention conceptions as the first phase of the larger project.

Data sources for this phase of the project included semi-structure interviews of each participant prior to the intervention and reflective activities/journals done at the beginning of the first meeting of the intervention. The interviews addressed, for example, their thinking about the meaning of authentic assessment and authentic assessment in mathematics, the nature and examples of mathematics authentic assessment task, what would be assessed by a mathematics authentic assessment task, and mathematics authentic assessment task observed during their field experience. All of the interviews were audio recorded and transcribed. Written journals or responses to prompts included their thinking about what is: a problem/task, a rich mathematics task, a good mathematics assessment task, and an example of a rich task for assessment in an elementary grade. The participants were also given eight mathematics tasks from Smith, Stein, Arbaugh, and Brown
Data analysis involved an approach in which the data were coded to identify themes in the participants’ thinking that represented their conceptions of authentic assessment and authentic assessment tasks. Using open coding, the data-analysis team initially scrutinized the transcripts to highlight statements that indicated the participants’ conceptions (e.g., their views, beliefs, knowledge) of authentic assessment and assessment mathematical tasks. The data analysis team consisted of the researcher who did not conduct the interviews and two research assistants, working independently. They categorized the coded information based on common characteristics of each participant’s thinking. They compared the categories among them and revised or eliminated categories based on disconfirming evidence upon re-examination of the data. They then compared the resulting categories across the participants to identify common themes in the participants’ conceptions of authentic assessment and assessment tasks. Again, they obtained agreement among them in order for a theme to be considered as representative of the participants’ individual and collective conceptions. They then used the Authentic Intellectual Quality criteria to examine these themes and explanations to determine the ‘quality’ of the participants’ conceptions based on our perspective of authentic assessment and authentic assessment tasks. A task-analysis scheme consisting of items of the Authentic Intellectual Quality criteria with a 4-point scale for each item was used to analyse their tasks. Scoring was done individually and collaboratively by the data analysis team to arrive at consensus. Finally, the participants’ scores of levels of cognitive demand of the set of tasks they analysed were compared to the suggested levels of Smith and Stein (1998).

**Results on Prospective Teachers’ Conceptions**

We provide key findings of the participants’ conceptions of authentic assessment and authentic assessment tasks in teaching elementary school mathematics.

**Conceptions of authentic assessment**

As one participant explained “authentic means real …real evaluation of where students are at with their math understanding and knowledge.” This is representative of how participants made sense of ‘authentic’ in conceptualizing authentic assessment. However, three themes emerged regarding their conceptions of authentic assessment: use to teacher, use to student, and the task.

**Use to the teacher.** All of the participants, to various degrees, viewed authentic assessment based on the use or purpose for the teacher. For example, they viewed it as: allowing the teacher to see, gauge, or understand what students actually know (e.g., kinds of ideas they can generate, their true understanding, what they learned from what was taught, their true comprehension of the topic, their thinking behind it), how the student is learning, where students are in their learning, and
whether students can apply the concepts and actually get some correct answers. It gives the teacher a broad, clear picture of the students’ understanding; a complete picture of what the students understand and their level of comprehension.

A few participants were more specific about the purpose of the assessment in relation to the teachers’ action. For example, “It allows them to learn what motivates their students, find their interests, then design work to then actually get proper data. … It is the first step to finding something that you can measure objectively.” It is “finding out and connecting the material, the subject matter to the student in the best way possible.” It is about “understanding how the student is learning … so that I know where I need to target for my students.” One participant explained that it was about the teacher being authentic and giving the students clear direction on exactly what their expectations are and what they are assessing.

**Use to the student.** Most of the participants also considered authentic assessment based on use or purpose for students and what they are expected to do. For example, it allows students: to really show what they learned during a lesson, to engage with the material in their way, to apply their mathematical learning, to take the information and actually apply it elsewhere, to actually demonstrate knowledge or understanding and show it in some way or in a variety of different ways, to seeing the real-world value of what they are doing, and to see the value in what it is they are trying to learn.

**The task.** All of the participants related authentic assessment to a real, real-world, or real-life task, application or situation, but a few felt that this did not have to be the case always. However, they varied in the purpose of the tasks. Most viewed it as a way to determine what the students know or what they are able to do and what they do not know or cannot do. For a few, it was about providing fun and motivation for learning the mathematics. More about their conception of tasks is provided in the next section.

**Conceptions of authentic assessment tasks**

As noted above, all of the participants associated “real world” (e.g., real-world example, application, concept, task, or problem) with varying interpretations as the central feature of authentic assessment tasks. However, they held different interpretations of authentic consisting of the following themes, presented in order of high to low dominance in their collective thinking.

**Real-world situations/examples.** The participants described these tasks as: having an authentic or real context, being a scenario or real-world problem where you are using mathematics, something you would really do outside of school, an actual task where students would use the mathematics context in real life, real-world kind of problems for kids, and involving “real-world problems that students will or may possibly very possibly encounter in their lives and will have to use their academic skills to circumvent.”
**Relevant.** Many of the participants considered relevance to be central to the tasks. For example, the task should be relatable and meaningful to students, should relate to students’ lives or could be applied to students’ lives, and should be interesting or of interest to students. However, they viewed this more as a means to encourage learning.

**Hands-on.** Some of the participants considered “hands-on” as being important to the task with a focus on the use of manipulatives. For example, the task should involve some sort of manipulatives, something they could touch and feel, some hands-on approach. A few noted that the task should also be something they could actually be doing (e.g., the real-life occupation or situation that uses that exact math concept). However, while they explicitly considered hands-on as features of assessment tasks, implicitly they were really viewing it as a means of instruction to engage students in learning the concepts.

**Represent learning.** Some of the participants considered the tasks as a way for students to represent their learning. For example, the task should “provide opportunity for students to display their learning in a meaningful way for the teacher to really see the whole process of what they know and what they don’t.” It should involve “demonstrating all the steps along the way, like what they’ve learned in the classroom”, “showing in many different ways about how they get that [answer]” and a “variety of ways to represent learning.”

**Mathematics problem/project.** Many of the participants directly or indirectly considered the tasks to be projects or word problems. For example, the task is: “A project that would bring the entire unit together … something that would draw from everything … from all different topics.”; “A project that’s the entire term that covered all the different concepts.” It is also a mathematics problem with many steps; a problem to apply their mathematical learning; a problem that they may have never encountered before – word problems; a problem that “don’t necessarily … [have] a desired end point that’s all the same.” One participant also considered the relationship to the solver:

> If I want to start at this y coordinate and end at this y coordinate, what equation would get us to do that? And it’s all about, like, ok, well, I have to come up with that myself. It’s very much just like self driven and not just like here’s an equation just plug in the values.

A few others felt that the problems should “mirror learning” and focus on “what they’ve been taught, with maybe some extensions to see if they can apply it” or require “using all the little things learned to solve a bigger problem (e.g., a project).” A few also thought the task could be about the mathematics concepts in the world. For example:

> The different ideas about fractions that we might find in the world as well as decimals. How many of the window panes have transparent versus opaque glass? Can we time ourselves running and then translate that into seconds and decimals?
Individualized/Personalized. A few participants thought that for the task to be authentic, “it should be individualized” and provide “opportunity for every child to find some aspect of what’s happening that they feel successful in.” It should be individualized to personally relevant and real-world interest of the student. The task should also allow for “a more personal or distinct individual presentation of the information [content]; there is an opportunity for every child to find some aspect of what’s happening that they feel successful in.”

Inter-disciplinary. A few participants emphasized tasks that had “connections of different subjects [school disciplines]” or “cross-curricular connections.” Some of these conceptions of the participants were influenced by examples they observed in their field experiences that included open tasks. However, these examples were not used as assessment but instructional activities for students to learn or practice the concepts. Examples of these tasks are about: (1) Students going to a store with a cookie recipe and had to find the most economical way to pay for their cookies. (2) A problem involving people getting on and off a bus with students explaining how they would know how many are on or off. (3) A project on gardening that included figuring out how many square meters the garden was. (4) Activities in which the teachers used candies to teach the concepts, for example, learning about mean, median, mode. (5) An activity in which students used unifex cubes to build a tower, then broke it in two at any point and subtracted the number of cubes in each part. (6) Students going for a walk in the neighbourhood, where each child counted something different, then they created a bar chart and computed percentages and fractions. (7) An activity involving making a budget for decorated cookies for an end-of-school year celebration

These were the examples the participants identified as examples of authentic assessment tasks during the pre-intervention interviews. However, they were also asked to create tasks that were rich mathematics tasks that could be used for assessment. Table 1 shows representative examples of tasks they created during the first session at the beginning of the intervention that they considered to be rich mathematics tasks and their thinking of why they are good assessment tasks. These examples are representative of how they conceptualized worthwhile mathematics and assessment tasks. For all of them, the richness of the tasks was the “real-world” context. For some, it also involved multiple solutions or multiple steps. But most of the tasks were routine-oriented problems that mimicked real-world situations as a context for the task and lacked depth in terms of the Authentic Intellectual Quality criteria.

As the examples in Table 1 indicate, the participants’ conceptions of what made these problem good assessment tasks for students included that they allow them to show: their thinking, if they understand the concept, if they know and understand the steps to get the answer, areas of their understanding and misunderstanding, if they can explain or demonstrate their thinking, and multiple answers.
<table>
<thead>
<tr>
<th>Tasks</th>
<th>Why good assessment tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td>If a baker bakes 3 cakes each day and the baker works for 5 days, how many cakes would they have? Show two ways to solve this problem.</td>
<td>Gives insight to not only if they know the answer but that they know and understand the steps to getting the answer.</td>
</tr>
<tr>
<td>Sally had 10 cookies. She gave 2 to Sam and ate 2 more. Steve had two more cookies than Sally, but he ate 4 cookies in total. How many cookies do Sally and Steve now have?</td>
<td>It would identify areas of understanding, and more importantly, misunderstanding that students have.</td>
</tr>
<tr>
<td>If you have 3 pieces of candy that you want to share with friends, but six friends, how can you be fair about sharing your candy evenly?</td>
<td>Students … can explain their thinking and demonstrate using manipulatives</td>
</tr>
<tr>
<td>A new community is being built and the builder needs to know how many houses can fit in one area. The land area is 5 km² and each house is 20 m²? How many houses can fit?</td>
<td>By asking them to show their work and explain the reasoning behind their answers … [shows] who understands the concept versus who does not.</td>
</tr>
<tr>
<td>Design a garden that has an area of 10 m². How much fence do you need?</td>
<td>Shows if the student understands perimeter and area; many different ways the student can show it</td>
</tr>
<tr>
<td>Janet goes to her neighborhood candy store with $4.75 in her pocket. She wants to purchase some candy for herself and her two friends. Gumballs cost $.10 each, licorice costs $.25 each, and lollipops cost $.50 each. What does Janet purchase at the candy store, how much does it cost and how much change does she take home?</td>
<td>Students show their mathematical thinking and multiple answers could be given.</td>
</tr>
</tbody>
</table>

Table 1: Prospective teachers’ examples of tasks for assessment

Following their creation of these tasks, the participants were given a set of tasks to determine and rank the level of cognitive demand of each, with level 1 being lowest and level 4 being highest. They were most challenged to identify level 4 tasks because of their focus on context. For example, this task was intended to be level 4 (doing mathematics, Smith and Stein, 1998).

Think of a real-life situation that describes the following problem 287/14 =. Write the problem and then solve it. (Smith et al., 2004)

Three of the participants viewed it as level 4 based on the context, for example, “students can transpose the problem into the context of their own lives,” “getting
students to think of applications of this problem,” “students show how mathematical understanding can be applied in daily life,” “prompting the student to recall a relevant real-life situation.” The other participants viewed it as level 1 ( memorization) mainly because it lacked appropriate context, for example: “An authentic task would be the opposite. Using a real-world application and getting students to derive the equation from it.” “There is such little context behind the problem,” “students are not performing an authentic task, but describing one,” “does not ask students to relate the problem in a personal way.”

**Discussion and Conclusion**

The findings indicate that the prospective teachers held limited understanding of authentic assessment and authentic assessment mathematics tasks based on our perspective of authentic assessment and the Authentic Intellectual Quality framework. This suggest that they are not adequately equipped to select, create and use authentic assessment in their teaching without support to help them to develop more depth in their conceptions of it.

Compared to the Authentic Intellectual Quality criteria, the prospective teachers’ conceptions included real-world connection, but their interpretations of it did not always attend to the authenticity of an out-of-school real-world task. In most cases, their interpretations indicated instrumental use of the real-world context or situation as a way for students to see the use of the mathematics concept being learnt. They also considered application and problem solving but for the most part these involved a focus on numerical computations and procedural knowledge. There was little or no consideration of levels of cognitive demand or levels of competencies associated with the tasks or the other Authentic Intellectual Quality criteria.

There was no indication that the prospective teachers drew on the STEM course to conceptualize authentic tasks, but they did make some meaningful connections to formative assessment from the assessment course. However, while the assessment course enabled them to develop initial understanding of formative assessment that was evident in their conceptions of authentic assessment, it did not allow them to conceptualize authentic assessment and tasks with enough depth that is needed to apply them to practice. This suggest that on their own, they were not able to transfer those experiences or theory to meaningfully conceptualize authentic assessment in teaching mathematics. However, they did draw on their field experiences regarding what they observed in their mentor teachers’ classrooms, but while in some of these classrooms the students were actively engaged in open-ended activities or projects, the tasks were not being used by the teachers as authentic assessment tasks. Thus, the participants made no distinction between the instructional and assessment use of tasks.

The study provides further evidence to support the need for special attention in teacher education research to determine effective ways of helping prospective
elementary school mathematics teachers to develop meaningful and useful knowledge of authentic assessment and mathematics tasks. A promising way to improve their knowledge will require engaging them in clarifying, extending, and reconstructing their conceptions of assessment and coherence among assessment, curriculum, teaching and learning, which our intervention is intended to do.

References


This study examines the content and pedagogical knowledge of pre-service elementary grades teachers. Our working hypothesis is that a teacher’s mathematical content knowledge influences her or his mathematical beliefs that in turn affect the decisions they make in their instructional activities. Our particular focus is to examine how the teachers’ knowledge of fractions and proportional reasoning impacts the views they hold about mathematics teaching and learning. While proportional reasoning is a major topic more often addressed in the middle and secondary grades, we agree with researchers who trace its source to the earlier grades when fractions and part-whole relationships are introduced (Ball, 1993; Carpenter, Fennema and Romberg, 1993; Pitkethly and Hunting, 1996). The study looked to examine how the teachers’ conceptual understandings of fraction and part-whole relationships inform their pedagogical decisions in problem situations that involve proportional reasoning. Specifically, the paper focuses on 1. Examining connections that teachers make between their mathematical content and pedagogical knowledge; and 2. Documenting these connections in illustrative examples.

Keywords: mathematical knowledge, pedagogical knowledge, mathematical beliefs

Proportional reasoning is “an important integrative thread that connects many of the mathematics topics covered in grades 6-8” (NCTM, 2000, p. 216). The ability to reason proportionally was a hallmark of Piaget’s distinction between concrete levels of thought and formal operational thought (Van De Walle and Lovin, 2005).

According to Lamon (2007), proportional reasoning problems involve situations in which quantities are compared and transformed according to particular mathematical operations. In order to solve these problems, students must utilize their knowledge of rational numbers and fractions to “reason up or down” in order to find a solution (Lamon, 2007, p. 648). For example, consider the following Laundry Detergent problem from Lamon (2007): If a box of detergent contains 80 cups of powder and your washing machine recommends 1¼ cups per load, how many loads can you do with one box?

In solving the problem, a student might reason that since 1¼ cups will wash one load, $1\frac{1}{4} + 1\frac{1}{4} + 1\frac{1}{4} + 1\frac{1}{4} = 5$ cups will wash 4 loads. So we must have $16 \times 5 = 80$ cups will wash will wash $16 \times 4 = 64$ loads.

Related research

Content Knowledge: Proportional Reasoning

While typically studied as reasoning that develops in grades 6-8, children’s difficulty with proportional reasoning in the context of conventional fractions in
grades 3 and 4 is noted in the mathematics education literature (Ball, 1993; Carpenter, Fennema and Romberg, 1993; Pitkethly and Hunting, 1996). For example, Ball (1993) reported that third-grade children systematically misinterpret traditionally notated fractions (e.g., ¾), and estimate that fractions with larger denominators are quantitatively greater than fractions with smaller denominators (e.g., 4/6 < 4/8). Relatedly, studies report that children as young as 5- to 6-years of age can successfully solve slightly modified proportional reasoning problems (Sophian and Wood, 1997). Children become capable of solving simple analogy problems during the preschool years (e.g., Gentner, 1977), and researchers have noted that proportional reasoning is a quantitative form of analogical reasoning, in the sense that both conceptual analogies and quantitative proportions require analysis of the relations between relations.

Since proportional reasoning has its sources in the early grades, a goal of this study was to assess the proportional reasoning knowledge of pre-service elementary grades mathematics teachers to determine how well they would be able to provide their students with the necessary mathematics foundation for solving proportional reasoning tasks.

**Pedagogical knowledge: Assessing student understanding**

The connection between a teacher’s content and pedagogical knowledge is difficult to isolate and study (Schoenfeld, 1994). For example, we would not expect pre-service mathematics teachers to use their content knowledge to think about all of the possible solutions that their students might demonstrate. Having teachers examine student work to develop an understanding of students’ mathematical thinking has been encouraged as a way for teachers to demonstrate content expertise in the context of instructional settings (Driscoll and Bryant, 1998; Moon, 1997; Brown and Clark, 2006).

Specifically, this approach allows teachers to identify and analyze the fundamental mathematics content and processes used by students thus providing them with a basis for making evidence-based conjectures about students’ mathematical understandings. Hence, the current study made use of tasks that required the teachers to consider hypothetical problem solutions of students solving proportional reasoning tasks.

**Goals and purposes**

The goal of this study is to examine connections between the content and pedagogical knowledge of pre-service elementary grades teachers in the context of solving proportional reasoning tasks. Specifically, the study looked to examine how the teachers’ conceptual understandings of fraction and part-whole relationships inform their pedagogical decisions in problem situations that involve proportional reasoning.
Methodology

Participants

The teachers who participated in this study (N = 4) were enrolled in a sophomore level mathematics course, entitled, MATH-2340 Number Concepts and Relationships, taught by the first researcher. The participants comprised a diverse subset of the class in terms of both ethnic and mathematics performance characteristics. The four participants consisted of one white male (Matt), one Latino male (Mario), one white female (Katelyn), and one African American female (Katherine). All names are pseudonyms for the actual participants.

MATH 2340 is a three-hour course that sits within a sequence of courses required for pre-service middle grades and elementary grades mathematics concentration majors. The course investigates integers, rational and real numbers, and examines conjectures and intuitive proofs in number theory. The pre-service elementary mathematics concentration teachers participating in the study had more formal mathematics than typical elementary grades teachers. Specifically, the teachers took on average nine more semester hours of mathematics than is required for elementary grades majors. Our rationale for this sample was to build on the work of researchers who have studied elementary pre-service teachers having advanced mathematics preparation (Thompson, 1992; Phillip, Flores, Sowder and Schapelle, 1994).

Measures and instrumentation

The teachers participated in a series of 4 tutorial interviews: In Interview 1 we asked the teachers a set of questions designed to find out about their formal background in mathematics as well to probe some of their beliefs and attitudes about mathematics teaching and learning (Table 1). In Interviews 2-4, the teachers solved a variety of content and pedagogical problems (Table 2).

| Tell me about your mathematics background in K-12 and college. |
| What are the attributes of a successful mathematics teacher? |
| What are the attributes of a successful mathematics student? |
| If you could change one thing about current mathematics classrooms and teaching, what would it be? |

Table 1: Introduction Questions
Solve mathematical content problems involving fractions and proportions

| Task 1: Order the fractions 4/7, 7/13, and 14/25 on a number line. |
| Task 2: Find a fraction between 5/6 and 11/12 |

Solve pedagogical problems based on hypothetical student work

| Task 3: The Magic Algorithm |
| o Find a fraction between 5/6 and 11/12 |
| o Suppose one of your students, Christine gave 16/18 as an answer. |
| o Is Christine’s answer correct? |
| o How do you think Christine got that answer? |
| o How would you discuss her solution with the class? |

| Task 4: Classroom Ratio Task (adapted from LMT, 2005) |
| In Mrs. Calabrese’ class the ratio of boys to girls is 4 to 5. If there are 12 boys in the class, how many students total are in the class. |

   Erin and Sean responded:

   **Erin:** In a class with 4 boys and 5 girls, the fraction of boys is 4/9, so I can solve the proportion \( \frac{4}{9} = \frac{12}{x} \).

   **Sean:** The way to represent a ratio like 4 to 5 is by using the fraction 4/5, so I started with \( \frac{4}{5} = \frac{12}{x} \).

Please comment on each student’s method.

| Table 2: Sample Content and Pedagogical Tasks Used in the Study |

The content problems (Tasks 1 and 2) provided a measure of their knowledge of operations with fractions while the pedagogical tasks (Tasks 3 and 4) provided examples of hypothetical student work on fraction and proportion reasoning tasks and asked the teachers to assess the students’ work.

**Data collection procedures**

The class instructor interviewed the teachers as they completed the tasks. Each interview lasted approximately one hour. All interviews were videotaped.

During the interviews, the teachers were encouraged to describe their formal experiences as mathematics students and verbally self-report their solution strategies as they completed the content and pedagogical tasks.
Data consisted of the teachers’ verbal and written work generated as they completed the tasks, and the interviewer’s field notes. Written transcripts were generated from the teachers’ video protocols.

**Data analysis procedures**

To analyze data, we summarized the teachers’ formal mathematical experiences and then examined the teachers’ work in Tasks 1-4 to identify and classify the various strategies they used to complete the tasks. We looked for common themes across the written work and classified the strategies accordingly.

**Results**

The results are reported as follows. First, we provide an overview of the teachers’ mathematical experiences and beliefs. Second, we briefly describe the teachers’ work on the content tasks (Tasks 1 and 2). Third, we summarize and illustrate the teachers work on the pedagogical tasks (Tasks 3 and 4).

**Mathematical experiences and beliefs**

The teachers’ mathematical experiences and beliefs are summarized in Table 3.

<table>
<thead>
<tr>
<th>Name</th>
<th>College Math Courses; Highest math completed</th>
<th>Class Grade</th>
<th>Attributes of Successful Math Student</th>
<th>Attributes of Effective Math Teacher</th>
<th>One change you would make in current math classrooms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matt</td>
<td>3 Calculus</td>
<td>A</td>
<td>Understand pure numbers and number operations Can make connections</td>
<td>Knows the material Knows where the math comes from Is open to alternative problem solving approaches</td>
<td>De-emphasize the textbook and have students provide more explanations</td>
</tr>
<tr>
<td>Mario</td>
<td>2 Calculus</td>
<td>B</td>
<td>Can think outside the box to solve problems in different ways</td>
<td>Must maintain open mind to students who do something different Must always look to what will interest the students</td>
<td>Implement more activities in the classroom. Students are more to likely to be able to remember and apply concepts that interests them</td>
</tr>
<tr>
<td>Katelyn</td>
<td>2 Pre-Calculus</td>
<td>C+</td>
<td>Understands the available resources and</td>
<td>Must be able to relate to all of the students</td>
<td>Provide more resources for teachers</td>
</tr>
</tbody>
</table>
uses them as needed | Must be open to a variety of approaches |
--- | --- |
Katherine 3 Calculus A Can think critically to solve math problems Can explain why a particular solution works Must be able to relate different approaches to solving a problem Must be approachable to all students Pursue a more holistic approach that focuses on making connections

Table 3: Overview of the teachers’ mathematical experiences and beliefs

The teachers’ responses comprise an interesting range of views about mathematics teaching and learning. For example, Matt and Katherine consistently remarked on the importance of the teacher having a strong grasp of the mathematics in order to help students see connections among various topics and the importance of helping their students to think critically to solve problems.

Matt: As a teacher, my goal is not to teach them math but to teach them how to solve problems. If ideally my students could do their HW and teach themselves, then I could bring them back together and discuss what they know about it.

Katherine: I think it is important for the teacher to have a holistic approach. So a need for students to learn connections between functions and equations. The textbooks never teach it that way – it never connects the pieces.

In contrast, Mario and Katelyn emphasized the importance of the teacher teaching the basic operations and thus providing for students a strong foundation upon which to build increasingly abstract concepts. Though a bit more focused than Matt and Katherine on teaching ‘the mathematics’ that students need, they each held some very interesting views about what their students would need to be successful.

Mario: I think it is important for students to learn basics and then explore many different problems. If they do not explore, they will not learn. Exploration is the key!

Katelyn: The basic operations are so important. That is not all you learn but they are things I learned in grades 1 and 2 that I am still using.

**Mathematical content knowledge**

Three of the four teachers used equivalent fractions to solve the content tasks. Of these, only Matt used prime factorization to find the least common multiple of the denominators of the equivalent fractions, commenting, “they will need this approach (finding the Least Common Denominator) later on”.

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Only Katelyn was unable to use equivalent fractions to complete the tasks. She was the only teacher who used a calculator to compute answers.

**Pedagogical knowledge**

The teachers’ solutions to Task 3 (Magic Algorithm Task) and Task 4 (Classroom Ratio Task) are summarized in Table 4.

<table>
<thead>
<tr>
<th>Name</th>
<th>Task 3: Magic Algorithm Pedagogical Task</th>
<th>Task 4: Classroom Ratio Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matt</td>
<td>Acknowledge the student’s contribution as a possible new method (and would “do some homework before next class to find out why it works!”)</td>
<td>Correct solution, that both students’ approaches could be used</td>
</tr>
<tr>
<td></td>
<td>Work to develop an activity around the new method</td>
<td>Viewed student solutions as equally appropriate to solve the problem</td>
</tr>
<tr>
<td>Mario</td>
<td>Acknowledge the student’s contribution as awesome!</td>
<td>Initially incorrect, thinking that only Erin was correct</td>
</tr>
<tr>
<td></td>
<td>Shows how thinking differently can pay off</td>
<td>Eventually acknowledged that both students could be correct</td>
</tr>
<tr>
<td>Katelyn</td>
<td>Acknowledge the student’s contribution as a shortcut for solving the problem</td>
<td>Incorrect solution, that Erin was incorrect</td>
</tr>
<tr>
<td></td>
<td></td>
<td>After much reflection on her solution, she maintains that she is correct</td>
</tr>
<tr>
<td>Katherine</td>
<td>Acknowledge the student’s solution as interesting but quickly move on – it is haphazard and not mathematical!</td>
<td>Correct solution, that both students’ approaches could be used</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Prefers Erin’s approach because it yields the solution directly</td>
</tr>
</tbody>
</table>

Table 4: Teachers’ performance on pedagogical tasks

Given the teachers’ strong mathematics background, we found the teachers’ performance on the Magic Algorithm task to be somewhat surprising because only one of the teachers, Katherine, thought the student solution to be problematic in terms of conceptual understanding. Specifically, she stated that it was haphazard and had nothing to do with the more mathematically appropriate ways of thinking about the problem and commented that it will not work for harder similar problems.

**Katherine:** How about 1/3 and 2/3, so we get 3/6 = ½, yes it works. Let’s try 2/4 and 3/4, we get 5/8 and yes, it works. However, I would be hesitant for her to get into the habit of it because it would make me nervous – that is not how we add fractions! I would discourage her from doing it. Just because it works doesn’t
mean that it will always work. There is no understanding, it seems haphazard! I think it solves this general problem but will not solve the harder problem of finding one halfway between. It seems we are breaking math rules so that is not going to work.

More typical among the participants was acknowledgement of the student work as important because it works.

**Mario:** I think Christine is awesome! I would applaud her with the other students.

**Katelyn:** Christine has found a shortcut; yes, it is always good that students can see different ways.

The Classroom Ratio task was much more difficult for the teachers since it required them to assess the solutions of the two students, Erin and Sean. Matt started by solving the problem and developed the most complete solution of all of the teachers. Based on his solution, he was not at all concerned with the difference between the two student solutions:

**Matt:** Okay, let me solve it. Therefore, we have 4 to 5 boys to girls. Therefore, if there are 12 boys we get 15 girls. But it doesn’t ask that. It asks for total, total is 27. You would have 15 girls and you add the two. I think they both work. He needs to add his up and she gets the total number. Sean’s is equivalent to mine. Erin did boys to total students so she got it right away. That is okay, Sean just needs to add them up in the last step.

In contrast, the other students did not first solve the problem; rather they commented directly on the students’ solutions and thus struggled more than Matt in solving the problem. For example, Mario was only able to solve the task after much effort:

**Mario:** I think that neither Erin nor Sean is correct. Erin has the wrong ratio. I would just do this in my head. I would know if it is 4 to 5 For every 4 there are 5 if you have 12 then 4/12 = 5/x, what, that doesn’t work either. I guess 4/12 = 5/x, 4x = 60 divide by 4, is it 15. So I guess that does work. If it is 4 to 5 students …. No, Sean would be right. Sean is right but it doesn’t gibe with the number of students. I see what Erin is doing. So, yes, Erin is right also.

Katherine acknowledged that both students could be correct but that she preferred Erin’s approach because it yielded the solution directly. Finally, Katelyn never ventured from her initial intuition that Erin was incorrect.

**Katelyn:** What she wrote, the ratio is 4 boys and 5 girls and she set it up, there are 12 boys, she did not put the boys with the boys or something weird. I thought it is 4/5 = 12/x, so Sean is right. I think she is wrong because 4/9 …. the fraction of boys, I do not know where she got 9! Did she add 4 and 5? Did she think it was the total? That is what I think. She is confusing fractions and ratios.
Conclusions

We must be careful not to conclude too much from these findings but some observations are noteworthy. First, the teachers’ ideas about teaching and learning were very rich and certainly confirm the importance of elementary grades teachers having a strong background in the mathematics they will teach (Schoenfeld, 1994; Thompson, 1992). Second, we are reminded that even with advanced mathematics preparation, teachers can be successful in many different ways. While we observed two of the teachers, Katherine and Matt appeared to be more focused on the importance of students developing as autonomous problem solvers, Mario and Katelyn’s emphasis on teaching ‘the mathematics’ by no means suggests that they would become rigid teachers. Both teachers stated the importance of the teacher providing challenging tasks to their students and being vigilant for the diverse problem solving approaches that their students might demonstrate. Finally, the results remind us of the importance to be able to understand the intricate connections between the teachers’ mathematical beliefs and the pedagogical views they hold.

References


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**ANALYSIS OF PROBLEMS WITH FIGURES IN PRIMARY MATHEMATICS**

*Lucia Csachová and Mária Jurečková*

**Abstract**

This paper focuses on successfulness of 10 and 11-year-old pupils in specific types of mathematics problems – problems with figures in the Slovak nation-wide testing in mathematics T5 and on some examples of such problems. On the base of obtained results, it seems that pupils have difficulties with some problems of such a type because they are not able to read data from figures in the assignment. On the other hand, some selected problems are easy for pupils because figures (e. g. illustrations) help them to solve problems.

**Keywords:** problems with figures, data visualization, nation-wide testing of mathematics

**Introduction**

In school mathematics, there are problems, whose assignment is not only in text format, but the text is completed with non-text elements (such as an illustration, scheme, chart, table or diagram) containing problem-related data. These problems are named *problems with figures* (Csachová, Gunčaga and Jurečková, 2017).¹ To solve them mean that the solver has also to read the data from the non-text part of the assignment and process it. Although working with tables or graphs is already part of primary mathematics, pupils consider some of problems with figures to be so difficult that they do not solve them at all. A similar situation has already been appeared with 15-year-old pupils (Csachová and Jurečková, 2018).

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¹ The term “figures” was also used by A. L. Mesquita in her article (1998) as a synonym for the representation of a concept or situation in geometry.
In the past and present many cases have been known, in which accidentally or intentionally distorted data in visualized form were provided (e.g. so-called misleading graphs). For that it is important to teach pupils to read comprehensive data from created charts, diagrams or other visual figures, to create simple or more complex examples of such figures and to evaluate different ways of visualization of data and to critically access them.

**Theoretical background, theoretical framework**

It should be emphasized that problems with figures play an important role in mathematics education. Correct data visualization and its reading from figures is very important for solving problems from different areas. Incorrect display or reading of data can significantly affect pupils’ understanding of issues, and cause misconceptions or blocks. “Reading” data from figures in problem’s assignment is especially important for two activities, namely the identification of objects in the figures and relationships between them.

The figure in item’s assignment can complement a text as an *illustration* (or sketch, topological scheme) when it does not meet the exact proportions specified in the assignment (angles, length of lines, ratios of lengths or perpendicularity), and the solver must use his / her imagination in addition to theoretical knowledge (Mesquita, 1998). In another case, the figure can appear as an *object*, which already satisfies the proportions and it is possible to infer other geometrical properties.\(^2\)

The figures in assignment of mathematical problems are very connected with data visualization. The role and meaning of data visualization in mathematics learning is described e.g. in (Rösken and Rolke, 2006), (Zimmermann and Cunningham, 1991) or (Arcavi, 2003) and (Bishop, 1989).

**Methodology**

Problems with figures are very often occurred in primary mathematics. In Slovakia this is apparent, for example, also in standardized nation-wide testing of mathematics T5 (in the next T5). T5 is designed for 10 and 11-year-old pupils at the beginning of lower secondary education in Slovak school system (the 5\(^{\text{th}}\) grade) and is regularly carried out by the National Institute for Certified Educational Measurements in Bratislava (NICEM) from 2013. The main aim of this nation-wide testing is to objectively compare the performance of individual pupils in the subjects selected for testing.\(^3\) Moreover, based on the results, the next aim is to obtain a general view of the performance of pupils at the exit of the first

\(^2\) The figure as an illustration can be the obstacle for some pupils.

\(^3\) Testing T5 is annually administered in the following subjects: Mathematics, Slovak language and literature, Hungarian language and literature (for pupils with Hungarian language as the mother tongue). NICEM prepares also another nation-wide testings in Slovakia – T9 for 15-year-old pupils and the external part of a school-leaving exam.
level of education (ISCED 1) and monitor the level of the readiness of pupils for further study (ISCED 2).

According to the State Educational Programme for primary education (ISCED 1) test items from mathematical part of T5 are divided into five content units:

- **U1 – Numbers, variables, operations with numbers,**
- **U2 – Relations, functions, tables and graphs,**
- **U3 – Geometry and measurement,**
- **U4 – Combinatorics, probability and statistics,**
- **U5 – Logic and reasoning.**

Every year 30 test items are included in T5 to verify the pupils’ ability to use mathematical thinking for solving various problems. 20 of them are open (a short numeric answer is required) and 10 items are closed (a multiple-choice with four offered options and just one of them is still correct). The numbers of items for each content unit are almost the same every year.

We focused on the data from T5 in the selected period of last three years 2016 – 2018. The source of the used data is the anonymized database provided by NICEM. It contains information about the results of T5 of 136.712 pupils of the 5th grade of lower secondary schools across Slovakia in this period. There were 45.299, 45.062 and 46.351 participated pupils in these years gradually.

The research questions which we interested in were:

- **What is pupils’ successfulness of problems with figures in Slovak nationwide testing T5?**
- **What is pupils’ successfulness of problems with figures in comparison with content units U1 – U5 in T5?**
- **Which problems with figures do pupils successfully solve and which not?**

In the following section we focus on the quantitative and qualitative analysis of obtained data.

**Quantitative analysis of the problems with figures**

In this section we present the statistical analysis of the results of T5 in mathematics in period 2016–2018. Besides the overall evaluation, we also pay attention to the results achieved in content units U1 – U5 and in problems with figures. Statistical analysis was carried out in the statistical program IBM SPSS and all charts in the section are the outputs of this program.

In order to avoid differences in the level of complexity of test items in each year, we have created common groups of all test items for individual content units over a given period. So that the first chart in Figure 1 presents the mean successfulness in T5 (in %) and mean successfulness for individual content units U1– U5 (in %) in the period 2016-18. The highest mean successfulness in T5 was achieved in the area U1 – Numbers, Variables and Operations with Numbers (the highest value
was achieved in this area every year – concrete values: 66.4 % in 2016, 70.3 % in 2017, 70.6 % in 2018). The significantly lowest value of results (44.4 %) was reached in the content unit U4 – Combinatorics, probability and statistics (in this content unit, the reached level was low in each year of this period). Interestingly, compared to the results of testing T9, see e.g. (Csachová and Jurečková, 2018), the content unit Geometry and measurement in T5 does not have a low success rate.

![Figure 1: Mean value (in %) of results in T5 and individual content units in period 2016 – 2018 (U1 – Numbers, variables, operations with numbers, U2 – Relations, functions, tables and graphs, U3 – Geometry and measurement, U4 – Combinatorics, probability and statistics, U5 – Logic and reasoning)](image)

![Figure 2: Mean value of results (in %) in selected areas: problems with figures (figures), Geometry and measurement (U3) and total successfulness of T5 in individual years](image)
The number of problems with figures is not stable, it ranged from 14 to 19 out of 30 test items in period 2016 – 2018. This type of mathematical problems is not only of the geometric nature, such test items belong to all content units. The chart in Figure 2 presents how is the mean value of gained points (in %) in T5, in a content unit U3 and in problems with figures developed in individual years. The total success of results in T5 was not very different in first two years – 62.3 % in 2016 and 64.7 % in 2017, but in 2018 there is a decrease – 59.3 %. In 2018, we can see a rapid decrease in a success also in U3 and in problems with figures.

Qualitative analysis of selected problems with figures

Previous results suggested that pupils have significant difficulties in some mathematical problems containing figures in assignment. In this section, we show selected problems with figures from T5 (every item is marked ORDER NUMBER / YEAR).

In T5, items with a mathematical context but also with a real-life context are included. Due to the age of tested children, the second group of items predominates. In Figures 3a, b there are two problems with figures from content unit U3 – Geometry and measurement with a mathematical context. Since the solution of the both problems are based on the simple use of conceptual knowledge – the relation between radius of a circle and its diameter, it would be expected more success. We assume the low successfulness of the both mentioned problems could be caused also by its abstract, mathematical context, that is not interesting for pupils of this age.

Figure 3: Figures from test items: a) 11/2016 (success rate 41.2 %): The circle k with the centre S has a radius of 3 cm. The circle m with the centre Z has a radius of 2 cm. The circles k and m touch at point C. Determine the length of the line XY in centimetres. (Notes: The picture has not right dimensions.), b) 10/2017 (success rate 54.1 %): The picture shows the square ABCD. The length of the side AB is 6 cm. Inside the square there is a drawn circle k with the centre S. Determine the radius of the circle k in centimetres.

As examples of real-life context problems with figures, we present following two test items in Figures 4 and 5, both aimed at using the relation between the length
of the sides and perimeter of a rectangle. In Figure 6 there is a supposed solution of one test item of them.

Figure 4: Figure from test item 09/2016 (success rate 50.3 %): Ondrej built a fence around the rectangular parcel. He used 38 m of netting to block the parcel. The longer side of the parcel was 12 m long. Calculate the length of the shorter side of the parcel. Record the result in metres.

Figure 5: Figure from test item 08/2017 (success rate 39.3 %): The father divided the parcel of rectangular shape for his sons into three parts that were of the same size and shape. The figure shows the division of a parcel. Samo and Marek agreed that they would merge their parts together with a single mesh. How many meters of mesh to fence their lands needed Samo and Marek together? Note: The mesh is shown in the figure with a black-grey dotted line.

Figure 6: Supposed solution of test item 08/2017 (10-year-old boy, the 3rd class); the figure is an environment in that the problem is solved

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4 In 2018, there was also a test item 16/2018 concerning the perimeter of a classroom with a rectangular shape. The test item had similar assignment as the item 09/2016.
At the end of this section we would like to point to two problems with figures which successfulness was high. In the Figure 7a there is a problem from financial literacy (such tasks are occurred in textbooks), in the Figure 8 a problem focused on using of the multiplication table. These both problems have a real-life context and seem to be easy. The solution by 7-year-old boy shown in the Figure 7b proves it (the boy wrote addition of the values of the given banknotes and gradual addition). During the interview, this boy said about the problem with flour bags in the Figure 8: “I was thinking about why there are three big bags and six small ones... two small ones must be a big one, so that the small bag must weight 4 kilograms ...”. We suppose that these problems were formulated in that way to be interesting for pupils. The first task is close for pupils; the second one can be looked like a riddle.

Figure 7a): Figure from test item 07/2017 (with success rate 80.3 %): *Sisters Janka and Danka saved money as shown in the picture. How much more euros did Danka save as Janka? Figure 7b) and its solution (7-year-old boy)*

Figure 8: Figure from test item 17/2016 (success rate 71.2 %): *The same amount of semi coarse flour and fine flour were brought to the bakery. The semi coarse flour was in*
three large bags, each bag having a weight of 8 kg. The fine flour was brought in six smaller bags. Each bag of fine flour had the same weight. What weight did one sack of fine flour have? Give the result in kilograms.\(^5\)

**Conclusion**

Problems with figures seem to be an interesting group of school mathematics problems. The imagination and the ability to “see (in geometry)” are involved in solving these tasks, even though they are expected to be in the geometry field. Analyses show that some problems with figures are difficult for pupils because pupils are not able to “read” some data shown in the figures. As we encounter such tasks in real life, it is necessary to teach pupils to read these data. We would like to focus on the detailed analysis of the problems with figures and their solutions, on reading data from scheme, pictures, tables or charts. Then we want to use this analysis to educate future teachers.

Except it the results of this nation-wide testing seem to us as a good basis for determining the strengths and weaknesses of school mathematics.

**Acknowledgement:** The paper was prepared with the support of the grant VEGA1/0079/19 “Analysis of critical areas in school mathematics and identification of factors influencing pupils’ attitude to mathematics”.

We would like to thank NICEM for providing research data and for assignments and figures of mentioned test items (www.nucem.sk/sk/merania/narodnemerania/testovanie-5).

**References**


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\(^5\) An interesting feature of this item is that its assignment could be purely textual, without a figure. But adding this figure to a text assignment makes the problem more active.
PERCEPTION OF MATHEMATICS ANXIETY AMONG PROSPECTIVE TEACHERS WHILE SOLVING WORD AND NONSTANDARD PROBLEMS IN MATHEMATICS

Radka Dofková

Abstract
Math is generally perceived as a difficult and unpopular subject. We can confirm that even Teacher Education Faculty students often perceive math with fear or even resistance. Aversion to math then leads to systematic evasion of some of its areas, such as for example word or nonstandard problems. Thus these were at the focus of a pilot research study among students of teaching at elementary schools. A total of 63 of the full-time form of study at the Teacher Education Faculty of Palacky University in Olomouc partook in the research. A modified version of the Abbreviated version of Mathematics Anxiety Rating Scale (A-MARS) was used. The goal was to establish perceived level of mathematics anxiety of prospective teachers in terms of solving word and nonstandard problems. The results in the above mentioned areas were further compared based on the students' pertinence to the respective study groups.

Keywords: mathematics teaching, mathematics anxiety, teacher, mathematics, word problems, nonstandard problems

Theoretical Framework
According to statistical estimates, approximately 20% of the population suffers from more or less serious psychological or physiological symptoms connected to the feelings of anxiety in confrontation with mathematical problems (Ashcraft and Kirk, 2001). Perceived anxiety connected to avoiding areas of mathematics causes extensive problems across education levels and everyday life of those affected (Eden, Heine and Jacobs, 2013).

Defining Mathematics Anxiety
Originally, the phenomenon of mathematics anxiety was called mathemaphobia. It was formulated by a teacher who found various striking emotional reactions on his students' faces while solving mathematical problems (Gough, 1954). In the 1970s the following definition of this phenomenon was accepted “[...] a feeling of

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tension and anxiety that interferes with the manipulation of numbers and the solving of the mathematical problems in a wide variety of ordinary life and academic situations” (Richardson and Suinn, 1972, p. 551). In a wider context, authors Jain et al. (2009, p. 240) described mathematics anxiety as the consequence of “an inability to handle frustration, excessive school absences, poor self-concept, internalized negative parental and teacher attitudes toward mathematics, and an emphasis on learning mathematics through drill without “real” understanding.” The description of causal factors of math anxiety is provided by Devine et al. (2012) who identified the variables connected to the development of math anxiety into three groups: influence of the environment, intellectual aspects, and personality predispositions. Influence of the environment includes negative experience of students with the class collective or family, intellectual aspects denote their level of cognitive abilities, and personality predispositions denote concepts such as self-esteem, self-conception, attitudes, self-confidence and learning (Eden, Heine and Jacobs, 2013).

An interesting moment might be the fact that some individuals suffer from math anxiety independently of their real competences (Ashcraft, Krause and Hopko, 2007; Hembree, 1990). Based on negative experience or random failure individuals then feel unsubstantiated anxiety, the level of which may increase and reach tremendous dimensions. A consequence of perceived math anxiety then is their inability to solve mathematical problems, diminished interest in math, avoiding math-related subjects, a limitation in the choice of colleges connected to the possibility to choose one's future career, and negative feelings of guilt and shame (Ashcraft, 2002; Betz, 1978; Richardson and Suinn, 1972).

Unfortunately, students’ negative reactions to math are often subconsciously strengthened or even literally approved in various social contexts (Beilock et al., 2010). Thus avoiding situations connected to math is more a rule than exception, and that regardless of the students’ age (Ashcraft and Faust, 1994; Hembree, 1990).

**Word and Nonstandard Problems in Math**

Word problems and their solving are an inseparable part of teaching math, and there are a whole series of various classifications of math problems (e.g. Blažková, 2002; Novák, 2003). In respect to the focus of this article, characteristics of word and nonstandard problems will be specified below.

**Word problems** as such have a special status in school mathematics, and students’ relationship to it is usually ambiguous. By word problems we usually mean practical tasks in which a real situation is described using a natural (nonmathematical) language, which results in a problem (Nováková, 2015). Blažková et al. (2002, p. 4) states that “by word problems we understand such problems, in which there is a connection between data given and sought expressed by words.” The way from a real life situation to the corresponding mathematical
model is called *mathematization of a real situation*, and it determines the connection between the given information and the mathematical result we seek.

Didactics of mathematics does not clearly define the notion of a nonstandard mathematical problem, but the existing curriculum explicitly works with the notion of nonstandard application problems. They are considered an important part of mathematics education and characterized as such problems “*whose solution may be independent of the knowledge and skills of school mathematics up to a certain extent, but for the solving of which it is necessary to apply logical thinking*” (FE PEE, 2013, p. 29).

These problems have a pronounced motivational character. Their goal is, among others, to show school math as an interesting and attractive subject. At the same time, while solving mathematical problems, especially context word problems with suitable topics from environments known to the students, interdisciplinary contexts are used (Nováková, 2016).

Regular incorporation of nonstandard problems into didactic preparation of mathematics education of prospective elementary school teachers may be a suitable instrument for practicing the work of pre-graduate teachers with such word problems. The attempt to learn some typical strategies of solving word problems, which the students might use, and mistakes they could possibly make, may prepare prospective teachers for giving their students feedback regarding their performance, and thus eliminate simplified teacher assessment of their abilities and skills (Dofková, 2016).

It is important to implement word and non-standard problems in lessons for pupils with special needs because the effort to them has been a major society-wide phenomenon at the national as well as global level in the last 10 years, and is generally considered a “higher and successful” form of integration. In terms of the education process, “inclusion” means the right of all people to full education, not just partial and conditional education as it is the case with integration. Inclusive education means an arrangement of a normal school in such a way that the school can offer adequate education to all children regardless of their individual differences, the types of their special needs and the performance of the pupils in class. Naturally we consider high-quality teachers in mathematics to be an integral part of a high-quality education of pupils with special needs (Mlčáková, 2014).

The aim of the paper is to establish perceived level of mathematics anxiety of prospective teachers in terms of solving word and nonstandard problems.

**Methods**

**Research Instrument**

Within the inquiry the Abbreviated version of Mathematics Anxiety Rating Scale (A-MARS) was used. It was originally created to measure students’ anxiety while
studying mathematics disciplines (Plake and Parker, 1982). Based on a research study conducted earlier, whose goal was to identify weaknesses in scientific and didactic preparedness of prospective elementary school teachers, it was found that word and nonstandard problems occupy some of the top places (Dofková, 2016; Dofková and Kvintová, 2017). Therefore, a slightly modified A-MARS was used, besides others, and two items were added to it focusing on the level of anxiety while solving these problems. One was Items 17: State what level of anxiety you feel if facing a set of word problems you are supposed to solve, and the other Item 18: State what level of anxiety you feel if facing a set of nonstandard problems you are supposed to solve. In the final version of the scale there were 28 items, and the respondents assessed each item on a 5-point scale from “low anxiety” to “high anxiety”.

**Research Sample**

The research was conducted in the winter semester of the 2018/2019 academic year at the Faculty of Education, Palacky University Olomouc. The instrument was applied among prospective mathematics teachers enrolled in mathematical courses. The pilot research involved a total of 62 students. There were 43 elementary school teacher program (ESTP) participants and 19 elementary school and special education teacher program (ESSETP) participants.

Both of these groups went through the mathematical-didactic with some differences according to the focus point of their study. The contents of the courses are very similar: the didactic system of primary-school mathematics, the basic tendencies of the development of mathematics teaching with a focus on pupils’ motivation to learn, creativity, pedagogical constructivism, research-oriented teaching etc. The course for ESETP is more focus on the special needs of pupils in mathematics because the graduate should gain pedagogical competencies for the work of a fully qualified teacher and specialized teacher.

The questionnaire items were evaluated on two basic levels – overall results and results by field of study. Regarding these levels, the following research questions were formulated:

(RQ1) What is the overall evaluation of math anxiety while solving a set of word and nonstandard tasks among the respondents?

(RQ2) Are there any differences in prospective teachers’ evaluation by study groups?

**Results**

Prior to assessing the items that were subject to research, an overall analysis of results was conducted. The possibilities in individual items were grouped into three main groups – a low level of anxiety (possibilities “none” and “mild”), average level (“average”), and a high level of anxiety (“high” and “extensive”).
It was found that the respondents felt the lowest level of anxiety:

- While signing up for a math subject, and while checking their receipt after paying for goods in a shop (in both cases 85.48%),
- While observing an elementary school teacher solving a math problem on the whiteboard (77.42%),
- While buying mathematics lecture notes and other materials (72.58%),
- While listening to another student explaining math terms (70.96%),
- While entering the classroom before a math lesson (69.36%),
- And while opening materials to start working on an assignment (51.61%).

On the other hand, the respondents felt the highest level of mathematics anxiety:

- During the final math test (83.87%),
- One hour before a math test or when the teacher unexpectedly gives them a test during a math lesson (in both cases 82.56%),
- While solving the math part of an entrance exam (75.81%),
- When they are supposed to solve a lot of difficult math problems for the next lesson (72.58%),
- One day before a math test (66.13%),
- While writing a continuous math test, and while opening math lecture notes and seeing a page full of math problems (in both cases 59.68%).

From the above overview it is apparent that items 17 and 18 that were subject to research do not occupy leading positions of either low or high levels of mathematics anxiety. In Item 17 the average was 3.16 (average assessment prevailed), and standard deviation of 1.22. In Item 18 the prevailing assessment was also average (3.51), and standard deviation just slightly lower (1.18). While analyzing their mutual relationship it was found that there was no statistically significant difference between the assessment of these two items (correlation between the items was .643352 and p-value was lower than the specified level of statistical significance).

While assessing Item 17, it was found that no anxiety while solving a set of word problems was felt by 8.06% of the students, mild level by 24.19 %, average level by 30.65%, high level by 17.75% and extensive level by 19.35%. While solving a set of nonstandard problems (Item 18) no anxiety was felt by 6.45% of the students, mild by 12.9%, average by 29.03% and high and extensive level by 25.81%. (Figure 1).
Then the zero and alternative hypotheses were set in order to verify differences between the study groups.

\( H_0: \) Prospective teachers’ responses do not vary by their study group.

\( H_A: \) Prospective teachers’ responses vary by their study group.

The Fisher’s combinatorial test was used to calculate the p-value for each item, which was further compared to the chosen level of significance of .05.

In Item 17, 26.32% of the students of ESTP mentioned a low level of mathematics anxiety while solving a set of word problems, 52.63% average, and 21.05% high. 34.89% of the students of ESSETP mentioned a low level of mathematics anxiety, 20.93% average, and 44.18% high (Figure 2). The calculated p-value in Item 17 was .7109, which was a higher value than the specified level of significance of .05, so that it was not possible to reject zero hypothesis. We can say that there is not a statistically significant difference in the level of mathematics anxiety while solving word problems between the ESTP and ESSETP groups.

While solving nonstandard problems (Item 18) 10.52% of the ESTP students and 23.26% of the ESSETP students feel a low level of mathematics anxiety; 47.37% of the ESTP students and 20.93% of the ESSETP students feel average anxiety;
and 42.11% of the ESTP students and 55.81% of the ESSETP students feel a high level of mathematics anxiety (Figure 3).

![Figure 3: Level of mathematics anxiety while solving nonstandard problems per study groups](image)

The calculated p-value in Item 18 was .7016, which again was a higher value than the specified level of significance of .05, so that also here it was not possible to reject zero hypothesis. We can say that there is not a statistically significant difference in the level of mathematics anxiety while solving word problems between the ESTP and ESSETP groups.

**Research Summary**

The results showed that prospective teachers have a balanced assessment of mathematics anxiety that they feel while solving a set of word problems, with a slight tendency to a higher assessment of experienced mathematics anxiety. The result is probably influenced by a high percentage of average assessment and inaccurate specification as to the level of difficulty of the problems in question.

Mathematics anxiety in students solving a set of nonstandard problems was at a significantly higher level. More than a half of the students mentioned that they felt a high level of anxiety. This result is also confirmed by experience from leading didactics seminars. Students do not have experience with solving this kind of problem. This type of problems is new to them, and if they do have any experience with them, it is mostly negative.

While comparing the results of the different study groups it is apparent that mostly the ESTP students had a tendency to give average assessment – in both items it was approximately 50% of all answers. While assessing perceived math anxiety while solving word problems, the remaining number of answers was almost regularly divided into the remaining parts of the spectrum. While assessing anxiety while solving nonstandard problems, a high level of anxiety was most frequently mentioned.

In the ESSETP group the level of experienced math anxiety while solving word problems inclined more to the higher level, as experiencing mathematics anxiety
while solving nonstandard problems was marked as high by over a half of the students.

**Discussion and limits**

The research results have confirmed our expectations based on long-term practice in preparing prospective teachers. It has transpired that in compliance with Richardson and Suinn (1972) students feel the highest level of math anxiety in situations directly connected to the level of their scientific knowledge (prior to a math test or in its course). Students of teacher education, however, feel a lower level of math anxiety in wider contexts connected to math problems, as described by Jain et al. (2009), such as using math in everyday life or mere observation of or listening to math performances of other students or teachers.

Ambiguity of the term of “nonstandard math problem” used in Item 18 might be in certain circumstances considered as a limitation of the research. Despite the fact that the respondents were told the definition of the term within the math didactics course, its ambiguous meaning might result in the respondents’ having problems assessing this item. In respect to the research results we can assume that this fact did not affect the given research, however, it will be appropriate to focus on it in the next research project.

**Conclusions**

We can optimistically assume that negative perception of math and elimination of math anxiety in students can be positively influenced any time. Based on this assumption we try and influence prospective elementary school teachers. We believe that if students get the right study materials and the teacher is able to manage their potential and abilities in a suitable way, their attitude to math can be significantly changed. Thus in accordance with Korthagen (2011) and his strategy of successive steps, the starting point for us is the Vygotsky concept of zone of proximal development (Vygotsky, 1986). In our opinion we should give students in didactics seminars what they want in order for them to feel safe, and then gradually increase demands. Thus we solve various types of problems with our students at the elementary level so that they can be sure solving them, and thus modified their attitude to math. We do not decrease the amount of mathematics content, but aspire to achieve something bigger that stands above that – educating teachers who will be self-confident and have a positive attitude to their profession, and transfer this onto their students.

**Acknowledgement:** This article originated within the project *Teaching situations in mathematics education* (IGA_PdF_2019_007), and the project *Preparedness of math teachers to develop digital literacy of students* (IGA_PdF_2019_001).

**References**


A TEACHER TRAINING EXPERIMENT: USING CHILDREN’S
LITERATURE IN TEACHING MATHEMATICS AND SAMPLE
APPLICATIONS

Burcu Durmaz

Abstract

The aim of this study is to examine the effect of a teacher-training project, which was supported by Turkey Scientific and Technological Research Council of Turkey (TUBITAK). This project is called Using Children’s Literature in Teaching Mathematics and Sample Applications and was funded under Innovative Education Practices Support Program. It was conducted from 25th of June to 2nd of July in Turkey.
The study group of the research was made up of 16 classroom teachers and 16 secondary school mathematics teachers, total 32 teachers from different cities in Turkey. The applicants of the research were selected via random sample method. It lasted for 8 days and a lot of professionals from different fields carried out a lot of workshops about integration children’s literature and mathematics education in multidisciplinary ways. It has a semi experimental research design. From the findings of the research we can conclude that it had a significant and positive impact on teachers’ creativity fostering behaviors, curriculum enrichment skills and differentiation skills.

**Keywords**: children’s literature, mathematics education, teacher education

### Introduction and Theoretical Framework

Mathematizing is a process of inquiring about, organizing, and constructing meaning with a mathematical lens (Fosnot and Dolk, 2001). By mathematizing books commonly available in classroom collections and reading them aloud, teachers provide students with opportunities to explore ideas, discuss mathematical concepts, and make connections to their own lives. Mathematizing a read-aloud also provides students with opportunities to learn mathematical concepts in meaningful contexts (Raymond, 1995). Using literature to connect concepts with students’ experiences helps foster understanding and motivates students to learn (Bintz, Moore, Wright and Dempsey, 2011; Shatzer, 2008). These kind of connections are essential to making mathematics accessible and for helping students use literature and mathematics to make sense of their lives (Lo Cicero, Fuson and Allexsaht-Snider, 1999). In addition to these, the use of picture books and extension activities creates positive reactions, enjoyment, and a sense of confidence in children (Shatzer, 2008). After the books have been read, utilized, and discuss, teachers can choose to create extension activities that expand upon the issues in the literature (Whitin and Wilde, 1992). Rozalski, Stewart and Miller (2010) have found that using carefully selected thematic books, teachers can use literature to reach young people who are experiencing difficult situations. Teachers can use children’s literature to reach a child in a non-threatening way by reading literature that can help to teach math concepts and really connect to the mathematical understanding of the learner and at the same time not intimidate, threaten, or turn-off a child to mathematics like some traditional approaches may have in the past. Children’s and adolescent literature can be a beneficial way of teaching mathematics.

By using children’s literature math concepts are taught in the context of a story; integration mathematics and children’s literature incorporates integrated studies with reading, writing, speaking, listening, etc.; develops mathematical thinking; prevents math anxiety and creates a less math anxious classroom environment; allows for a variety of responses which is important for mathematical discussions; makes historical, cultural, and practical applications and connections (may make multiculturalism alive); may allow for the use of manipulatives as it relates to the story; the teacher can assess a child’s understanding by reading/questioning; may
lead to problem solving and active involvement from the context of the story and provides a shared experience for students and the teacher (Furner, 2017). Although using children’s literature have lots of benefits, it is important to be careful for a distinction between using literature in the teaching of mathematics. Teachers may use literature to introduce, teach, reinforce, and make important connection to many math concepts through the use of story books that hold mathematics themes. There are also books that may be used to assist students with helping them come to terms with things like fear of mathematics for example, the book, Math Curse (Scieszka and Smith, 1995), or the book, Counting on Frank (Clement, 1991), which can be used to help gifted math students to accept and respect their giftedness with mathematics (Cited in: Furner and Kenney, 2011). Reading children’s literature, fairy tales, and stories while teaching math concepts can allow students to invoke more creativity and employ their imaginations further while making important mathematical connections to their understanding.

In the literature there are some studies which were focused on the effect of this integration. For example in a study by Mink and Fraser (2005), a mathematics program in which children’s literature was integrated (from the kindergarten to the fifth grade) was evaluated in terms of the impact of children’s books on the classroom climate and attitudes and had positive feedback for these variables. Thatcher and Fletcher (2008) have found that not all teachers realize or see the value of using literature for teaching. Children’s literature books are wonderful to read as a shared experience with children as they teach mathematical concepts within the context of a story making the learning of mathematics more meaningful to the learner. When teachers use fairy tales and children’s literature in their classroom to teach math, they are allowing for creativity, imagination, and making connections for students better preparing them for a world which is ever advancing mathematically and technologically. Lastly Edelman (2014) concluded that being aware of or using the children’s books are not enough, prospective teachers are in need of more support. As a result of this study, Edelman suggests that the methodology (Special Teaching Methods) courses taught in the undergraduate programs of mathematics education should be reviewed and studies should be carried out in line with this need.

**Aim of the Study**

According to the mathematics education literature we realize that in service and prospective (student teachers) teachers are not good at integrating children’s literature with mathematics instruction. They have problems with using children’s books or popular science books in an effective way for teaching. For these reasons, we decided to serve a 8 day training on this topic for Turkish teachers. Aim of this study is to nurture the elementary school teachers’ and secondary school mathematics teachers’ creativity fostering behaviors and differentiation skills.
Methodology

For this purpose, a group of 32 people have been selected from elementary school teachers and secondary school mathematics teachers who work in public schools from different cities (especially disadvantaged regions) all around the country. The study group of the research were selected from 600 applicants via some questions about children’s literature and mathematics curriculum integration, so in that study purposive sample methods were used. Some of the important selection criteria of the participants were work experience (>10 years) and knowledge about children’s books with mathematics context. Data collecting tools were Creativity Fostering Teacher Behaviour (Soh, 2000) and Self Assessment of Differentation Practices (Dacey and Gartland, 2009) scales. This study had an semi-experimental design with pre-test post-test group. Within the scope of this study data collection tools were applied as pre and post tests. Data obtained from these tests were analyzed with SPSS 20.0.

Findings

At first Cronbach’s Alpha reliability coefficients of the data calculated which were obtained from Creativity Fostering Teacher Behaviour (CFTB) were .955 for pre test, .961 for post test. After testing the reliability of the obtained measurements, the normality of the distribution of the data were tested in order to select the appropriate analyzes. The results of the analysis regarding the normality of pre-test and post-test scores are given in Table 1.

<table>
<thead>
<tr>
<th>Test</th>
<th>Number of Applicants n</th>
<th>Mean $\bar{X}$</th>
<th>Standard Deviation sd</th>
<th>Kolmogorov Smirnov</th>
<th>Shapiro - Wilk</th>
</tr>
</thead>
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<td>Pre</td>
<td>32</td>
<td>226.031</td>
<td>23.56</td>
<td>.200</td>
<td>.125</td>
</tr>
<tr>
<td>Post</td>
<td>32</td>
<td>246.063</td>
<td>17.57</td>
<td>.148</td>
<td>.044</td>
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</tbody>
</table>

Table 1: Test of Normality

Since the number of samples was over 30, the Kolmogorov Smirnov test was examined. The distribution of the scale scores obtained from both the pre- and post-test were found to be normal because of the significance level of $p>.005$. Therefore, t-test for related samples, which is a parametric test, was used to compare pre-test posttest scores. The test results of the pre-test post-test scores were given in Table 2.

<table>
<thead>
<tr>
<th>Test</th>
<th>N</th>
<th>Mean $\bar{X}$</th>
<th>S</th>
<th>sd</th>
<th>t</th>
<th>p</th>
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</thead>
<tbody>
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<td>23.56</td>
<td>31</td>
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<td>.000</td>
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<tr>
<td>Post</td>
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<td>246.063</td>
<td>17.57</td>
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</table>

Table 2: t test results
According to the table, pre-test and post-test scores were examined to determine whether there was a difference between the mean scores of the scores obtained from the scale applied before and after the study. There was a significant difference between the mean \( t(32) = -5.751, p < .01 \). The effect size calculated by the test result \( d=1.02 \) indicates that this difference is quite high. This shows that the activities implemented within the scope of the study have a statistically significant effect on teacher behaviors that encourage creative thinking.

In that study we used an another scale which is for differentiation practices of the teachers’. The Cronbach’s Alpha reliability coefficients of the Self Assessment of Differentiation Practices Scale (SADP) are .874 for the pre-test and .836 for the final test, respectively. After testing the reliability of the obtained measurements, the normality of the distribution of the data was tested in order to select the appropriate analyzes. The results of the analysis regarding the normality of pre-test and post-test scores are given in Table 3.

<table>
<thead>
<tr>
<th>Test</th>
<th>Number of Applicants</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Kolmogorov Smirnov</th>
<th>Shapiro - Wilk</th>
</tr>
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<tbody>
<tr>
<td>Pre</td>
<td>32</td>
<td>226.031</td>
<td>23.56</td>
<td>.200</td>
<td>.068</td>
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<tr>
<td>Post</td>
<td>32</td>
<td>246.063</td>
<td>17.57</td>
<td>.200</td>
<td>.900</td>
</tr>
</tbody>
</table>

Table 3: Test of Normality

Since the number of samples was over 30, the Kolmogorov Smirnov test was examined. It is seen that the distribution of the scale scores obtained from both the pre- and post-test is normally distributed as \( p>.005 \). Therefore, the associated samples t-test, which is a parametric test, was used to compare pre-test post-test scores. The test results of the pre-test post-test scores are given in Table 4.

<table>
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<tr>
<th>Test</th>
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</table>

Table 4: t test results

According to the table, the effects of the activities performed during the study on the self-efficacy of differentiation skills were positive, because there was a significant difference \( t(32) = -5.422, p < .01 \). The effect size calculated as a result of the test \( d=0.958 \) indicates that this difference is quite high. This shows that the activities carried out within the scope of the study have a significant effect on self-efficacy of differentiation skills.
Conclusions and implications

This study has positive impact on both mathematics education and children’s literature fields. According to the research findings we can see that the activities implemented within the scope of the study have a statistically significant effect on creativity fostering teacher behaviors and differentiation practices of the teachers. At the end of the study all teachers wrote their children’s books via Web 2.0 tools such as storyjumper and storybird, so they can use them for their mathematics lessons in different ways.

We can put forward that this teacher-training program had positive impacts on target behaviors and skills linked with teaching practices. Although the study group of the research have 10 years or more of professional experience, the majority of teachers did not have a high awareness of children and popular science books with mathematical context. Both individual and group stories that they wrote, it was seen that there are expressions that can cause important mathematical misconceptions. Teachers, before learning about how they can integrate such books with lessons, they still lack of knowledge on effective integration even if after the study. The most important skill for them is the selection of books with appropriate content for the mathematics classrooms.

In the literature, it is seen that the studies of the Children’s Literature courses, which remain very limited in the contribution of teacher candidates to teaching skills. In these studies, teachers stated that they could not acquire skills about how to integrate children’s books with their lessons. Teachers who graduate from the undergraduate programs (for example, elementary mathematics teacher, science teacher, etc.) who are not taught this course are more aware of the courses they are taught in the undergraduate program. Such integration, which may indicate the art dimension of the STEAM approach, can be taught as an elective course in undergraduate programs or emphasized in special teaching methods I and II courses. Lastly, however the applicants of the study had learned and worked a lot on children’s literature and mathematics integration they still have difficulties on an effective integration and misconceptions. Further studies and workshops are still needed.

Acknowledgement: This study was obtained from a project, which was funded by TUBİTAK (The Scientific and Technological Research Council of Turkey).

References


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**RAISING PRE-SERVICE ELEMENTARY SCHOOL TEACHERS’ AWARENESS OF THE IMAGES OF MATHEMATICS AND MATHEMATICIANS AS THEY ARE DEPICTED IN POPULAR CULTURE**

**Donna Ericksen and Tibor Marcinek ★**

**Abstract**

The paper discusses the results of a classroom study focused on capturing and documenting the effect of a course designed for pre-service elementary teachers to raise their awareness of the images of mathematics and mathematicians in popular culture.

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Analysis of various student writings and suggestions for more comprehensive future studies are provided.

**Keywords:** stereotypes, mathematics in popular culture, pre-service elementary teacher education

**Theoretical Background**

The problematic relationship between stereotyping and learning mathematics has been documented in the literature. Most of the research in STEM education focuses on studying the effect of gender stereotyping on students’ success and opportunities to learn (Cvencek, Meltzoff and Greenwald, 2011; Lindberg, Hyde, Petersen and Linn, 2010; Nowicki and Lopata, 2017). Research that tries to unveil the impact of stereotypes of mathematics and mathematicians is, however, much sparser. Such stereotypes are associated with an unfavorable image of mathematics and mathematicians, and have been shown to decrease use and appreciation of mathematics. Picker and Berry (2000), asked 215 K-8 students from five different countries to draw a scientist. Their study revealed that many children hold stereotypical images of scientists (such as they are male, wear eyeglasses and lab coats.) The problem with such stereotyping is that many students do not want to fit the stereotype and do not seek out and value the opportunities to learn mathematics.

Moreau, Medick and Epstein (2010) state, “The data suggests that far from being the uncritical viewers described by the media, students revert to the default image of the mathematician available in popular culture due to a lack of alternative resources to draw on, rather than because they are unaware of the cliched nature of such images.” Moreover, Dick and Rallis (1991) found in their research that one of the strong influences on students and their future career choices were those students’ teachers, hence underscoring the importance of raising future teachers’ awareness of stereotypes and provide them with knowledge to mitigate their negative effect. Mendick and Moreau’s research (2014) found that students should not just draw from the images that they see in popular culture, they need to “negotiate” the meaning of these images so that they can “tell their own stories.” Educating future teachers about the prevalence of these stereotypes and helping them to reflect on them in terms of their own teaching might aid in preparing teachers for such negotiations when they enter their own classrooms.

**Rationale for the study**

Thirty-three students in an introductory college level Algebra class were asked to complete the sentence “Mathematicians are….”. The responses these students gave, with their frequencies were as follows: Smart/Intelligent/Brilliant (17), Crazy/Insane (4), Logical (2), Dedicated/Hardworking (2). The keywords Important, Interesting, Intense, Weird, Unique, Complex, and Brave occurred once. These same students were further asked why they chose these words.
Students who chose to describe mathematicians as smart usually responded that mathematics was hard, and mathematicians know how to “follow the rules.” The students who chose crazy or insane gave comments such as: mathematicians “like numbers,” and “are the only people on the face of the planet who like to sit and look at numbers all day.” These same students were also asked to complete the sentence “Mathematics is…” Again, similar themes came through with keywords hard/challenging/difficult (15), confusing (4), interesting (2), easy (2), horrible/brutal (2). The keywords foreign, exact, fabulous, necessary, flexible, and intimidating appeared once. While more of the descriptions of mathematics as a subject were positive, the descriptions of mathematicians revealed many negative stereotypes that are often associated with mathematicians and mathematics in general. In fact, with the exception of the student who described mathematicians as “important they are everywhere in life” the remainder of the comments were overwhelmingly negative in that they supported the stereotype that ‘you must be smart to be a mathematician and like mathematics’. This information led the researchers to wonder what perceptions of mathematicians preservice elementary school teachers (PSTs) who were majoring or minoring in mathematics have and what might be the ways to address them in teacher education preparation courses.

A task similar to Picker and Berry’s study, to draw a mathematician, was presented to a group of 17 Central Michigan University PSTs majoring or minoring in mathematics who planned on teaching elementary school mathematics. Looking at their 18 drawings (one student drew two mathematicians), of the ones where the sex of the individual was apparent, 14 depicted males, 13 of the drawings showed the mathematician wearing glasses, while 9 of the mathematicians in the drawings were given crazy hair. One PST drew a dragon instead of a person. It was clear that many of these future teachers held the same stereotypes as seen in the Picker and Berry’s research.

These findings motivated the design of an elective course for preservice elementary school teachers majoring or minoring in mathematics with the goal of making them more aware of how mathematics and mathematicians are presented in popular culture. The goal of this paper is to present key features of the course and discuss the results from a classroom study focused on finding out if such exposure helps PSTs raise awareness of these stereotypes and think about how they as future teachers might be able to influence their future students’ perceptions about mathematics and mathematicians.

**Method**

Each week, the PSTs enrolled in the course were presented with mathematics as it appears in popular culture in the form of movies, cartoons, jokes, clothing, books, songs, poetry, televisions shows and television commercials. As part of the course assessment, PSTs submitted weekly journal reflections. These reflections centered on either the PSTs’ reactions to the media they were shown or listened
to, or the reflections centered on helping PSTs to think about these presentations in a broader context. Other reflections were based on an interview of friends and family, an interview of a mathematician and readings of research papers related to the course material. These reflections included questions probing PSTs’ perceptions of the value of these activities: What insights did you gain from doing this activity? What was your reaction to what you learned? De-identified reflections were qualitatively analyzed to note similar and different thought patterns in the PSTs enrolled in the course. Individual student reflections that mirror the class’ overall sentiments or typical reactions to the presentations and assigned work are presented.

Results and Discussion

One of the first activities involved having PSTs interview five of their friends or relatives who were non-education majors and ask them to give as many adjectives as they could to describe a person who would belong to each of the following organizations, “Band, Cheerleading, Drama Club, Football Team, Math Club and Honor Society. This activity was designed to give PSTs a first look at how widespread stereotypes of mathematicians and students good at mathematics are within society. A typical response is captured in the weighted word cloud in Figure 1. Larger font represents descriptors selected by at least half of the respondents.

![Figure 1: Typical descriptors for various student groups](image-url)

Figure 1: Typical descriptors for various student groups
As expected, most of the words that were used by the student’s friends or relatives to describe mathematicians and people who enjoy doing math fell right in line with many of the stereotypes that the PSTs would see as we began to look at the portrayal of mathematicians in popular culture. It was interesting that band members were often described with similar words as was used to describe the math club members. This raises an interesting question as to what other professions are plagued by stereotype depictions.

This is what one student wrote concerning what they had gained from the activity.

“...One student wrote concerning what they had gained from the activity.

“I gained a lot of insights about stereotypes. First off, I interviewed my mom and it was interesting to see how similar our answers were. My mom’s stereotypes have influenced the stereotypes I have. I learned that our parents play a big role in teaching us about other types of people, and we learn from them what is acceptable in society.”

While a second student responded:

“I began to wonder if students in the elementary and middle school grades are choosing their interests based on what they truly like or if they choose based on the perceptions of each club in media and schools. I can’t say that I was shocked by the results that I gathered from my interviews, but I feel that these descriptions of each group may be contributing in the downfall of band programs, math clubs, and drama clubs.”

These reactions show that PSTs in the course were beginning to reflect on where the origin of some stereotypes may come from and also the impact that holding these stereotypes might have on their future students’ desires to study mathematics.

One of the other activities of the course was to have each of the PSTs interview a mathematician. Professors in the Mathematics Department at Central Michigan University were asked if they would be willing to be interviewed by the PSTs in the course. This activity was particularly enjoyable for the PSTs as it gave them a chance to interact with the professors in the mathematics department in an informal manner. After completing the interview one student commented,

“How interesting [name removed] was both a wonderful and beneficial experience for me as a person, student, and future teacher. He cleared up some misconceptions about both mathematics as well as mathematicians by being both open and honest about his life and work. He showed me that mathematicians are just normal people; they have lives outside of work, hobbies, and families. He opened my eyes to some issues I never new [sic] existed. He made me aware of both the struggles and the joys I will face as a math teacher. This interview showed me that my job as a teacher is going to be much more then [sic] just helping students understand concepts; but rather to inspire students and show both the value of mathematics and its practicality.”
While these initial activities gave PSTs some background knowledge on how others might perceive mathematics and mathematicians and also a glimpse of how some real mathematicians might fit or not fit popular stereotypes, the bulk of the course was designed to open PSTs’ eyes to the portrayal of mathematics and mathematicians in popular culture and what direct and indirect messages about mathematics and mathematicians the media might be giving to the general public.

PSTs in the course were presented with many images of mathematicians as they are represented in popular culture. They viewed movies, television shows and TV commercials where mathematicians or mathematics were part of the central theme. They looked at jokes and comics about mathematics and mathematicians, and adult books and children’s books where mathematics or mathematicians were depicted. They listened to the Jimmy Buffett song “Math Suks” (Buffet, 1999), and a recording of the infamous “Barbie” uttering “Math Class is Tough” and they looked in catalogs to see what t-shirts with math messages were available.

As each cultural artifact (be it a comic, a television show, a movie or some other form of popular culture) was viewed, the PSTs would discuss what views of mathematics were presented (both positive and negative) of the mathematicians or mathematics in the artifact. PSTs would then be asked to write a journal reflection of their own personal response to what they had seen or heard. Along with giving their own reflection on what they saw depicted in the media, each student was asked how they might be able to change the perceptions of their future students if their students saw truth in a negative media portrayal, or how to build on the positives that were seen in the media.

In one exercise PSTs were given a comic that showed two teenagers playing guitar with a caption below that read, “I start every song by counting 1-2-3-4 because it reminds me of math, math depresses me and that helps me sing the blues.” (Glasbergen). One PST’s reflections included the following:

“A good portion of the population tends to become anxious when they must associate with math; this feeling may be eased to a sense of relief when reading through the newspaper and seeing a comic such as this one, giving the message of “don’t worry, you are not alone, everyone hates math”. A variety of math comics and jokes seen widely by Americans in various ways, shows that a struggle with math is common amongst the population, yet again allowing for a sense of relief for those people who may have thought they were the only one struggling.

In a nutshell, my comic is sending the message that “math is depressing” to its readers. Not only does the character say that he thinks math is depressing, but math reminds him of the musical genre ‘the blues’ which is classic soul music that may sometimes be sad. I’m sure that Jimmy Buffet could have easily made his “Math Sucks” song into more of a ‘blues’ genre which would fit perfectly with this comic. Many people will find this comic humorous simply because they can relate to it once reading it, thinking “ha-ha, yes, math is indeed quite depressing”. One thing
I really like about this comic is that it uses two male characters who think that math is depressing, hence because it is difficult and confusing, although stereotypically speaking, males are supposed to be good at math.”

By midterm, the PSTs had viewed many popular culture sources where images of mathematicians and mathematics were depicted. As their midterm assignment, the PSTs were asked to read and report on a research paper that might show how stereotypes could potentially influence the field of mathematics and mathematicians.

Topics included the effect of stereotypes in general, gender-related topics, stereotypes and the field of mathematics, the influence of parents’ stereotypes on their children (in areas such as help with homework and choosing careers) and what influence the belief in a “math gene” has on students.

This assignment was important as it bridged the gap between becoming aware of stereotypes and thinking about what positive or negative effect these stereotypes might have on their future students. Each student was able to readily relate their research paper to many of the sources of media that they had experienced in class. Each student was also able to express the positive or negative effect of the stereotype on their future students. They also were able to discuss how they could work to dispel the effects of this stereotype if it were negative or build on those stereotypes which might be positive. As one student summarized in his paper:

“This article is extremely important for mathematics educators to read, for we need to know that this stereotype is actually a major problem. If we cannot teach our student mathematics properly and without harm to their learning, then why are we even attempting to teach it in the first place?”

Although PSTs were asked to comment on each movie, television show, comic, children’s book or other media that was presented each week, at the end of the course PSTs were asked to create a chart of the various stereotypes that they found in each of the media sources (both positive and negative.).

The PSTs were also asked to provide evidence as to where they witnessed the various stereotypes (from what popular culture source) as well as a quote or give a visual description of what stereotype they had viewed or heard. While this information was important in terms of having PSTs recognize the prevalence of many of the stereotypes as well as the importance of evidence to back their claims, it is not importance in terms of particular stereotypes that PSTs became aware of in the images of mathematics and mathematicians that are being presented in popular culture. Figure shows the list of the stereotypes, both positive and negative, that the PSTs in the course noticed as appearing in various forms of popular culture.
<table>
<thead>
<tr>
<th>Negative Stereotypes:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Girls are bad at math</td>
</tr>
<tr>
<td>African American mathematicians are not as capable as white ones</td>
</tr>
<tr>
<td>Math does not help us in the future</td>
</tr>
<tr>
<td>People who are good at math wear nerdy clothes, pocket protectors, etc.</td>
</tr>
<tr>
<td>People who are good at math are “square”/bad with relationships/dating</td>
</tr>
<tr>
<td>It is normal/okay to be bad at math</td>
</tr>
<tr>
<td>Mathematicians wear glasses</td>
</tr>
<tr>
<td>People who are good at math are obsessed with it</td>
</tr>
<tr>
<td>Mathematicians are “know-it-alls”</td>
</tr>
<tr>
<td>Mathematicians have no friends/nobody likes them</td>
</tr>
<tr>
<td>Women who are good at math are harder to come by than men</td>
</tr>
<tr>
<td>Even parents/adults hate math/are bad at it</td>
</tr>
<tr>
<td>Women don’t get the same opportunities in the field as men do</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Positive Stereotypes:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematicians are viewed highly in society</td>
</tr>
<tr>
<td>Mathematicians can be found where we least expect it</td>
</tr>
<tr>
<td>Mathematics are not nerds/geeks</td>
</tr>
<tr>
<td>Mathematicians come in all shapes and sizes</td>
</tr>
<tr>
<td>Mathematicians always have good jobs/make a lot of money</td>
</tr>
<tr>
<td>Mathematicians are hard workers</td>
</tr>
<tr>
<td>Mathematicians are good at/have interest in other things other than math</td>
</tr>
<tr>
<td>Knowledge of math can help with many other things other than math tests</td>
</tr>
</tbody>
</table>

Figure 2: Positive and negative stereotypes noticed by the PSTs

This chart shows awareness of the presence of math in popular culture that PSTs observed while interacting with the various media forms. This begs the questions as to what impact this awareness had on the PSTs that were enrolled in the class? One student’s final reflection can shed some light on that question:
“As a future teacher, it is my job to give my students the tools to be the most successful, confident, educated, and well-rounded individual that they can possibly be. I believe that in our day and age, the media gets in our way in doing so. Our students are being influenced by what they see, hear, and read in the media. Until taking this class I hadn’t put much thought into how negative/destructive this influence can really be. After watching the movies and clips and studying the other forms of media such as t-shirts and songs, my eyes have been opened to how many stereotypes there are about mathematicians and mathematics. Although some of the movie portrayed positive stereotypes, I felt that the overwhelming amount of negative ones overshadowed them.”

Conclusion and Implications for Future Work

In their research, Mendick, Epstein, and Moreau (2007) found out that using popular culture in the mathematics classroom can open up mathematics to more people and such can be viewed as a component of pedagogy for social justice. Deliberate discussions of artifacts of popular culture can make positive relationships with mathematics available to more students and provide “alternatives to the current limited range of ways of relating to mathematics” (Mendick, Epstein and Moreau, 2007, p. 22).

This paper addressed how a course was designed to provide future elementary teachers with opportunities to learn about the images of mathematics and mathematicians that exist in popular culture and opportunities to reflect on how these images may be impacting their students’ learning and how they might combat these stereotypes when they become teachers. Still, there are several important questions that should be addressed by future research. To better reveal an added value of the course, a more detailed look at the stereotypes that the PSTs were aware of before taking the course and those that they recognized as a result of the course should be carefully documented.

Anecdotal evidence suggests that PSTs enrolled in the class became much more aware of the negative messages that exist in media. Many PSTs would demonstrate such an increased awareness in informal discussions: “I was watching TV and you would not believe what this person said about math” or make a similar statement about how someone commented negatively about math in the book they were reading. They mentioned how in the past they might have passively dismissed these comments but now were much more aware of them. Documentation of the changes in PSTs’ awareness of these messages as a result of the course should be done more formally to help measure the course’s impact. Similarly, now that these pre-service teachers are in schools, it would be interesting to see if they followed through on any of the ideas that they had for creating a more positive image of mathematics and mathematicians in their classroom. Figure lists some of their ideas.
This study illustrated an impact of the course on pre-service elementary teachers’ perceptions and awareness of typical images associated with mathematics and mathematicians. Further research should measure that impact and assess whether the effect is short-lived and limited to the course semester or has a more permanent impact on future teachers and consequently their students.

**References**


MATHEMATICS HOMEWORK

Jasmina Ferme and Alenka Lipovec

Abstract

Homework is a complex phenomenon whose effectiveness is influenced by many factors. We examined several characteristics of mathematical homework and their connections with pupils’ knowledge. The survey was conducted in Slovenia on a sample of 192 pupils of the first three years of the elementary education. Based on the obtained results, we conclude that the variables: a) the share of completed tasks and b) the optimisation of time when doing homework, are of special interest. They are in a positive correlation with each other; and additionally, they both are related with pupils’ self-reported mathematics grade. Based on these findings, we provide some suggestions for the school practice.

Keywords: completion, time optimisation, frequency, time, self-reported grade

Theoretical framework

Homework is defined as tasks assigned to students by school teachers that are meant to be performed during non-school hours (Cooper, 2006, p. 70). Empirical research that studies the relationship between achievements of students and homework gives inconsistent results (comp. Cooper, Robinson and Pattal, 2006; Fan, Xu, Cai, He and Fan, 2017). Most studies suggest a positive relationship (Fernández-Alonso, Suárez-Álvarez and Muñiz, 2015), but there are also studies that do not confirm that relationship or report even negative effects of some mathematics homework characteristics on pupils' mathematical achievements (De Jong, Westerhof and Creemers, 2000; Trautwein, Köller, Schmitz and Baumert, 2002). One of the reasons for the inconsistency of the results may lie in diversity of homework characteristics.

Teacher involvement in homework usually appears in two phases: when planning the scope and type of homework given to pupils, and by giving feedback to pupils

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When planning homework, teachers determine, among other things, the frequency and the time duration of a homework. In Slovenia, Podgoršek, Ferme and Lipovec (2017) reported that the vast majority (over 97% of respondents) of 4th grade pupils receive mathematical homework at least three times a week, most of them finish the single homework in 16 to 30 minutes. Additionally, the relationship between the frequency of homework and the achievements of pupils at TIMSS 2015 was low, but positive, while the time spent by pupils for a single mathematical homework was in a low negative correlation with pupils’ achievements at TIMSS 2015. Trautwein et al. (2002) also recorded similar results in a survey conducted in 1976 with seven-year-old pupils from Germany. Interestingly, Murillo and Martínez-Garrido (2013) in the results of a study with more than 5000 3rd grade pupils reported that the frequency, type, and the time spent on mathematics homework do not affect students’ academic achievements.

Teacher's responses to homework (for example, checking if the homework is completed) could also have an impact on students’ achievements. Núñez et al. (2015) reported that by the pupils’ reported teachers’ response to the completed homework is indirectly linked to academic achievements of pupils through the influence on pupil’s behaviour regarding homework assignments. Similarly, Walberg (1991) and Murillo and Martínez-Garrido (2013) reported positive effects of monitoring, reviewing and (immediate) correction of homework. On the other hand, De Jong et al. (2000) reported that the verification of homework assignments by teachers is negatively related to the achievements of pupils.

Beside teachers’ role in mathematical homework, there are several others factors that could influence effectiveness of mathematics homework. Several researchers studied the number of homework assignments completed by the pupils and it seems that it could be positively related to pupils’ achievements (Núñez et al., 2015; Núñez et al., 2015a; Cooper et al., 1998). Similarly, according to some studies (Núñez et al., 2015; Núñez et al., 2015a; Xu, 2011), the amount of completed homework is related to the optimisation/management of time doing homework (focusing on work while doing homework). Some also confirmed a positive relationship between the optimisation of time for performing homework and achievements of pupils (Núñez et al., 2015). However, no study, mentioned in this paragraph, was focused solely on mathematics homework.

**Research problem**

The aim of the research is to examine some of the Slovenian mathematics homework characteristics (especially those determined by teachers and pupils themselves) and to associate those characteristics with pupils’ knowledge measured by pupils’ self-reported mathematics grade. We have set forth the following research questions.
1. How frequent and time consuming is Slovenian mathematics homework according to pupils’ reports?

2. How do pupils perceive teacher responses (for example, checking homework)?

3. What is the percentage of mathematical homework completed by pupils?

4. Why is the homework not completed?

5. Are pupils disturbed by other factors while doing homework (how to pupils’ optimize their time)?

6. What is the relationship between some homework characteristics and pupils’ mathematical knowledge?

**Methodology**

We have used the methods of quantitative empirical pedagogical research. The survey was carried out based on questionnaires from a convenience sample of 192 6-8 year old pupils from Slovenia. The survey was conducted at the end of 2018. Data obtained from questionnaires were analysed with IBM SPSS Statistics 25; we used the following statistical tests: chi-squared test, t-test for independent samples, ANOVA and Spearman’s rank correlation coefficient test.

**Instrument**

Anonymous questionnaire consisted of several types of questions: a) questions about pupils’ basic data (class, gender), b) questions regarding self-assessment of pupils’ mathematical knowledge via grades and c) questions related to some characteristics of mathematical homework. The last type consisted of questions about the teachers’ role in mathematical homework (frequency, time duration and teacher's responses regarding homework) and questions related to the characteristics of the pupils’ homework performance (the share of completed tasks in homework, the reasons for unfinished tasks, and the level of time optimisation).

The teacher's responses to homework were determined by the level of pupils’ agreement with statements, summarized by Núñez et al. (2015). Pupils expressed the level of agreement with the statements on a five-level Likert scale, where 1 was a complete disagreement, and 5 a complete agreement with a single statement.

**Sample**

Participants were pupils from the 1st, 2nd and 3rd grades from Slovenia (N = 192). In Slovenia, first-graders, starting school, are 5 years and 8 months to 6 years and 8 months old. Share of boys in the sample was 41.7%, 58.3% of participants were girls. The structure of the sample is presented in Table 1.
In the continuation of this paper, we will use the abbreviation MHW – math homework.

First, we present the results regarding MHW frequency in Table 2.

<table>
<thead>
<tr>
<th>How often do you get a homework assignment in mathematics?</th>
<th>f (f %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>We do not get homework in mathematics.</td>
<td>0 (0.0)</td>
</tr>
<tr>
<td>Less than once a week.</td>
<td>13 (6.8)</td>
</tr>
<tr>
<td>Once or twice a week.</td>
<td>47 (24.5)</td>
</tr>
<tr>
<td>Three times a week.</td>
<td>76 (39.6)</td>
</tr>
<tr>
<td>We get homework at each mathematics lesson.*</td>
<td>55 (28.6)</td>
</tr>
</tbody>
</table>

*In Slovenia, math lessons take place four times a week in 1st and 2nd grade and five times a week in 3rd grade.

Results show that more than 67% of pupils receive math homework at least three times a week.

Table 3 shows the frequency of pupil responses to the question of the usual time duration of a single MHW.

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*Slovenian grades from the 4th grade are on the level 1 to 5 with 1 as non-passing grade and 5 as the best grade. During first three years, no numerical grading is present.

Table 1: Sample structure

Results

Table 2: MHW frequency
When you get a homework assignment in mathematics, how many minutes does it usually take to finish it?

<table>
<thead>
<tr>
<th>Time Duration</th>
<th>f (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 minutes or less.</td>
<td>88 (45.8)</td>
</tr>
<tr>
<td>More than 15 minutes, but less than an hour.</td>
<td>93 (48.4)</td>
</tr>
<tr>
<td>More than an hour.</td>
<td>10 (5.2)</td>
</tr>
<tr>
<td>I do not know since I do not do math homework.</td>
<td>1 (0.5)</td>
</tr>
</tbody>
</table>

Table 3: Time spent for completing single mathematical homework assignment

Almost one-half of the pupils (45.8%) finish a single MHW in less than 15 minutes.

Furthermore, we were interested whether the variables frequency of MHW or the time extensity and the pupils’ self-assessed grade are related. While this relationship cannot be confirmed in the first case (P = 0.583, $\chi^2 = 1.951$), there exists a relation between MHW time duration and self-assessed grade (P = 0.002, $\chi^2 (tr) = 15.145$). Differences in MHW time indexes (the average of time duration, where 15 minutes or less is coded as 1, and more than one hour is coded as 3) with respect to different pupils’ self-assessed grades are statistically significant (P = 0.000, t = 3.894). For pupils who reported grade 5, the time index is 1.46, while for the rest of the pupils the time index is 1.80.

Teacher’s response to MHW as perceived by pupils was determined based on the level of pupils’ agreement with the statements written in Table 4.

<table>
<thead>
<tr>
<th>Statement</th>
<th>$\bar{x}$, standard deviation SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>The teacher checks whether pupils have done their math homework.</td>
<td>4.60 (0.695)</td>
</tr>
<tr>
<td>Math homework is corrected in class to fix the errors.</td>
<td>4.35 (0.967)</td>
</tr>
<tr>
<td>The teacher gives students positive reinforcement when their math homework is done.</td>
<td>4.23 (1.114)</td>
</tr>
<tr>
<td>The teacher emphasizes the importance of completing the math homework.</td>
<td>4.56 (0.814)</td>
</tr>
</tbody>
</table>

Table 4: Teacher’s responses to homework

The results show that pupils’ agreement with all written statements is a relatively high, which implies that they are relatively sensible for the teachers’ response to their homework.
For each pupil, the level of perceptions of the teacher's response was calculated as the average of the individual's level of agreement with all four statements. Based on the results, we cannot confirm that pupils with different self-assessed grades in different extent perceive the teacher's responses ($P = 0.090, t = 1.705$).

Next, we consider the share of completed tasks of MHW. As is shown in Table 5, the majority of pupils (81.3%) usually complete all or almost all of MHW.

<table>
<thead>
<tr>
<th>Usually, how many tasks do you complete from the tasks in assigned homework?</th>
<th>f (f %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>None or almost none.</td>
<td>3 (1.6)</td>
</tr>
<tr>
<td>Approximately half of them.</td>
<td>33 (17.2)</td>
</tr>
<tr>
<td>All or almost all.</td>
<td>156 (81.3)</td>
</tr>
</tbody>
</table>

Table 5: The share of completed tasks of MHW

As the most common reasons for uncompleted MHW, the pupils state the following: “Because I cannot do it (I do not know how to solve tasks)” (21.9%), “Because I run out of time (due to other activities)” (16.7%) and “Because I forget to complete it” (16.7%). The tasks completion index was calculated as the average of the completed tasks, where we denoted by 1 none or almost none completed tasks and by 3 all or almost all completed tasks. The index is 2.69 for pupils with self-assessed grade of 4 or less, and 2.90 for pupils with self-assessed grade 5. The statistical test showed that task completion indexes are statistically different for pupils with self-assessed grade less than 5 and for those with self-assessed grade 5 ($P = 0.002, t = 3.112$).

The level of time optimization while doing MHW was expressed by the pupils through the answer to the question, written in Table 6.

<table>
<thead>
<tr>
<th>Are you, when doing homework for mathematics, distracted by other things (such as cell phones, talking to other people, television)? Do you think about other things, while doing homework for mathematics?</th>
<th>f (f %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I do not do homework.</td>
<td>1 (0.5)</td>
</tr>
<tr>
<td>Other things always distract me. I often think about other things.</td>
<td>19 (9.9)</td>
</tr>
<tr>
<td>Other things sometimes distract me. I sometimes think about other things.</td>
<td>95 (49.5)</td>
</tr>
<tr>
<td>While doing mathematical homework I think only about the homework. Nothing distracts me.</td>
<td>77 (40.1)</td>
</tr>
</tbody>
</table>

Table 6: Time optimisation of MHW
Only about 40% of pupils think solely about MHW while doing it, and are not distracted by other factors. On the other hand, almost 10% of the pupils’ report, that they are often distracted by other things and think about other things while doing MHW.

The time optimization and the share of completed tasks are related ($P = 0.000$, $\chi^2_{(tr)} = 21.176$) and this correlation is positive ($\rho = 0.304$, $P = 0.000$).

Related are also the variables time optimization and pupils’ self-assessed grade ($P = 0.008$, $\chi^2_{(tr)} = 9.730$). The task completion index and average pupils self-assessed grade (see Table 7) differ statistically for pupils with different time optimisation levels ($P = 0.000$, $F = 12.337$ for the task completion index and $P = 0.007$, $F = 5.051$ for average pupils’ self-assessed grade).

<table>
<thead>
<tr>
<th>Time optimisation</th>
<th>Task completion index (1-3)</th>
<th>Average pupils’ self-assessed grade (1-5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Other things always distract me. I often think about other things.</td>
<td>2.42</td>
<td>4.29</td>
</tr>
<tr>
<td>Other things sometimes distract me. I sometimes think about other things.</td>
<td>2.77</td>
<td>4.54</td>
</tr>
<tr>
<td>While doing mathematical homework I think only about the homework. Nothing distracts me.</td>
<td>2.94</td>
<td>4.68</td>
</tr>
</tbody>
</table>

Table 7: Time optimisation

Discussion

The role of the teacher in planning MHW was studied via frequency and time duration of MHW, which were reported by pupils. More than 28% of students receive daily MHW and almost 40% receive MHW three times a week. The results are somewhat inconsistent with the study carried out by Podgoršek, Ferme and Lipovec (2017), respectively by Murillo and Martínez-Garrido (2013). Both namely report that more than 45% of 4th respectively 3rd grade pupils receive daily MHW. Lipovec and Ferme (2018) also report that more than 67% of pupils from Slovenia and 74% of pupils from Croatia who attend the last three grades (7th, 8th and 9th grade) of elementary education receive daily MHW. This raises the question of whether MHW are more frequent for the higher grades. From the point of view of time duration of MHW, we have found out that 46% of pupils finish MHW in less than 15 minutes and 48% of pupils need between 15 and 60 minutes. This result is not in a line with the research carried out by Murillo and Martínez-Garrido (2013), who report that less than 30% of pupils need less than 30 minutes for a single MHW. Also, Podgoršek, Ferme and Lipovec (2017) report that there
is only around 30% 4th grade students who finish MHW in less than 15 minutes. Since in Slovenia MHW is, beside reading, prevalent HW in early school years, we believe that we can refute the doubts of many teachers or the over-worried care of a parent that HW takes a lot of afternoon family time.

Based on the results of the research, math achievements and MHW frequency are not connected, what is in line with the study carried out by Murillo and Martínez-Garrido (2013). On the other hand, some other studies (Trautwein, Köller, Schmitz and Baumert, 2002; Fernández-Alonso, Suárez-Alvare and Muñiz, 2015; Podgoršek, Fermeand and Lipovec, 2017) showed otherwise; namely that the frequency of homework is positively related to student achievement. Our findings, which reveal the relation between MHW time extensity and self-assessed grade, and validated differences in MHW time indexes with respect to different pupils’ self-assessed grades, indicate the negative correlation between the mentioned two variables. The negative correlation between time duration of HW and achievement is also reported by several other studies (Trautwein, Köller, Schmitz and Baumert, 2002; Podgoršek, Ferme and Lipovec, 2017). As Cooper and coauthors (Cooper, Lindsay, Nye and Greathouse, 1998) argue, perhaps one of the reasons for the ineffectiveness of time-consuming tasks is that if pupils spend too much time on the selected content, they can start to bore, which results in a decline in achievement. We therefore suggest that when designing HW teachers should pay more attention to HW time extensity than the frequency of HW.

Teacher also play an important role in giving feedback to students about homework. Pupils largely perceive teachers’ responses (the level of concurrence on the five Likert scale with each of the claims is at least 4.23). We were not able to confirm the relationship between the level of pupils’ perceptions of the teacher’s response and the pupil’s self-assessed grade. Slovenian results are inconsistent with the research carried out in the Netherlands (De Jong, Westerhof and Creemers, 2000), where the results showed that only 15% of teachers regularly check if pupils did their homework, while 12% of teachers do not carry out this activity.

In addition, to the teacher's role in MHW, we were also interested to the pupils’ characteristics regarding MHW: the share of completed tasks and the time optimisation of doing homework. Over 80% of pupils in first three grades usually complete all or almost all of their MHW. The variables the share of the completed tasks and the pupils’ self-assessed mathematical knowledge are related. Based on statistical differences in completion index regarding pupils self-assessed grade, is seems that this correlation is positive. This is in a line with several other studies (Núñez et al., 2015; Núñez et al., 2015a; Cooper et al., 1998). This repeatedly indicated correlation suggests importance of the by pupils’ reported causes for uncompleted tasks. As the most common reasons for unfinished MHW pupils reported very different things (difficulty level, low time optimisation, bad study
habits). Murillo and Martínez-Garrido (2013) reported that the individualized homework is positively related to pupils' achievements; therefore, perhaps the first cause of non-completion of tasks implies the need for more individualized tasks. Better time optimisation could help with the second and third reason. The variables of time optimisation and the share of completed tasks are namely positively connected. The latter is in line with several studies (Núñez et al., 2015a; Núñez et al., 2015; Xu, 2011) and thus may give rise to the idea that poor time optimisation (poor focus on work) means less completed tasks. Therefore, it seems that better time optimisation of doing homework can help to increase the share of pupils’ completed homework and consequently positively affects pupils’ math knowledge. Considering that only about 40% of pupils think solely about HW while doing it and almost 10% of students report that other things distract them all the time, we suggest that primary teachers promote better time optimisation. This suggestion is in line with results of Muhlenbruck, Cooper, Nye, and Lindsay (1999) who state »teachers in early grades assigned homework more often to develop young students’ management of time«. An additional argument for promoting a better time optimization is in research that confirms the positive relationship between time optimization and pupil’s math achievements (Núñez et al., 2015).

Conclusion

Our research revealed some of the characteristics of Slovenian mathematical homework, but due to the complexity of the phenomenon, more in depth research will be needed to cover a wider spectrum of factors that can affect the effectiveness of MHW. Nevertheless, we believe that additional research is needed that focus solely on mathematics and analyses of completing MHW and time optimisation impact.

References


This study explores Turkish Grade 6 students’ image of mathematicians and their work, stated attitudes to mathematics, and perceived needs for mathematics. Data was collected using the Draw a Mathematician Test (DAMT). This article is based on the drawings where students depicted a mathematics teacher in the classroom that also presented the mode of instruction being used, through students’ eyes. Trends that emerged for this sample included, in the drawings, (a) the most common mode of instruction was Highly teacher-directed, (b) no evidence of group work or Highly
student-centred mode of instruction existed, and (c) a whiteboard and/or books were the most remarkable teaching resources in classroom portrayals.

**Keywords:** drawings, images, mathematics classrooms, teaching practices

**Introduction**

Students sometimes perceive mathematics as difficult and abstract with lots of formulas and rules that are unconnected with each other and irrelevant to their lives. These perceptions of students can affect their achievement in mathematics and may keep them from effectively learning mathematics (Boaler, 2015). Teachers’ usual request is for students to work harder or to be more engaged with teaching activities. Nevertheless, these requests would be meaningless without explicit strategies for achieving them (Bobis, Anderson, Martin and Way, 2011). For decades, research in mathematics education suggests that teaching different mathematical skills might require teachers to access a range of different instructional strategies (e.g., Schoenfeld, 1992; OECD, 2016). Some countries take this research to heart and construct their mathematics curriculum to necessitate or strongly suggest that teachers use “a variety of teaching strategies.” (OECD, 2016, p. 15).

In Turkey, mathematics and science educational policies promote the principles of constructivism and student-centredness. The current primary and lower secondary school mathematics curriculum highlights the importance of active involvement of students in learning processes. The curriculum suggests that teachers embrace various teaching strategies considering students’ individual differences and to use appropriate concrete materials and/or ICT technologies when relevant. Among others, the curriculum aims to develop students’ mathematical, basic science and technology, and digital competences (The Ministry of National Education, 2018). Students’ classroom experiences, however, have remained relatively unexplored through research.

A previous research, led by the author, explored a large group of 1284 lower secondary students’ (grades 6 to 8) images of mathematics through examining students’ drawings. Drawing on mathematics theories in related literature, the previous research focused on three particular aspects of the image of mathematics: students’ stated attitudes (Lane, Stynes and O’Donoghue, 2014; Sam and Ernest, 2000; Wilson, 2011), perceived needs for mathematics (Wilson, 2011), and views about mathematicians and their work (Sam and Ernest, 2000). As previously described by Picker and Berry (2000), the students’ drawings fell into two distinct groups: drawings where students depicted a mathematician at work, and drawings where students depicted a mathematician as a mathematics teacher in the classroom. This article presents the data regarding the latter group. The research questions asked were: Through the students’ eyes, in mathematics classrooms, (1) What are the modes of instruction? and (2) What resources are used? This will
allow us to understand classroom teaching practices from the students’ perspective that could inform teacher education and future research.

The modes of instructions

Depending on whether it is the teacher or the student who plays a main role in the learning process, instructional practices are often grouped into two types: teacher-directed and student-centred (Thomas, Pederson and Finson, 2001). Teacher-directed methods include explanation, demonstration, questioning, and giving examples and/or counter examples. Student-centred methods include group work, problem solving, student presentations, open-ended tasks, games, and peer learning (Bobis et al., 2011). For many years, mathematics teachers have been encouraged to employ student-centred teaching strategies (e.g., Utley and Showalter, 2007; OECD, 2016) rather than the traditional teacher-directed teaching styles (Utley and Showalter, 2007), or to use a blend of teacher-directed methods with student-centred ones to achieve variety in teaching methods (Bobis et al., 2011). Results show that teacher-directed teaching practices “increase students’ factual knowledge and their competency in solving routine problems but have no significant effect on their reasoning skills” (Bietenbeck, 2014, p. 143).

The reality of classroom practices, however, is often different. Sometimes, pre-service teachers envision a classroom that is more teacher-directed than student-centred (Utley and Showalter, 2007), and teachers express more student-centred beliefs in the teaching strategies (Isikoglu, Basturk and Karaca, 2009). Accordingly, in mathematics classrooms, students mostly experience a teacher-directed style of teaching (Picker and Berry, 2000); many students sit at desks, passively listen to the teacher who stands in front of the class and lectures, and knows the content and delivers it to the students (OECD, 2016). The drive to explore and integrate the use of current teaching methods into mathematics classrooms is imperative in providing excellent teaching and learning in the mathematics classrooms.

Drawings as a type of measure

In educational research, inquiring into individuals’ own conceptions of their educational experiences is vital (Haney, Russell and Bebell, 2004). Although classroom observations or questionnaires have been used in this research for some time, “there is considerable scope for the development of new methods and the wider use of established methods for qualitative studies.” (Fraser, 2014, p. 116). One of the available techniques to document conceptions of individuals about teaching and learning experiences is drawings (Gulek, 1999). Over time, the use of drawings as a measure of the perceptions of young students was found to be a valid (Losh, Wilke and Pop, 2008) and a less expensive alternative to systematic classroom observations (Haney et al., 2004). In mathematics education, the “Draw a Mathematician Test (DAMT)” (Picker and Berry, 2001) patterned from the “Draw a Scientist Test (DAST)” (Chambers, 1983) (see Thomas et al., 2001, for
a detailed review) has been used from early childhood to grade 12 level in many countries on different continents including Europe, the Middle East, Asia, and the United States.

Large-scale assessments such as TIMSS and PISA identify various aspects of school and classroom climate, but these surveys have not been able to identify types of teaching practices (Vieluf, Kaplan, Klieme and Bayer, 2012). Researchers in mathematics education have used DAMT as a way to evaluate teaching in mathematics classrooms (e.g., Pehkonen, Ahtee, Tikkanen and Laine, 2011). In this article, students’ (DAMT) drawings and writing are utilized to have information about their perceptions of the teaching and learning practices in mathematics classrooms and the resources used.

The study

The study from which this paper evolved was primarily qualitative and was conducted in Turkish schools in Ankara, Turkey. The DAMT was used (with permission) to collect data by a research team led by the author. DAMT combines drawings with written responses. The front page provides a rectangular area in which participants are asked to draw a mathematician at work. Open-ended items eliciting written responses are provided on the back of the sheet. Relevant to this study is the item: “Look back at the drawing you made of a mathematician at work and write an explanation of the drawing so that anyone looking at it will understand what your drawing means, and who the persons are in it.”

To ensure the clarity of the instrument and to decide the time necessary for completing it, we piloted the instrument with 130 lower secondary students at three schools not participating in the actual study. After the pilot, the DAMT was sent to schools by the respective district Directorate of National Education to maximize the response rate. In schools, teachers other than mathematics teachers provided directions to and collected data from the students. We chose to survey students in classes other than mathematics to eliminate a possible mathematics teacher effect. It took students approximately thirty minutes to complete the DAMT. The schools sent the data in a sealed envelope to protect participant confidentiality.

A convenience sample of 1284 students from twenty different lower secondary schools (grades 6 to 8), under the auspices of the Ministry of National Education, participated in the study. The schools were co-educational metropolitan schools located in the centre of the city. In this article, I present the grade 6 student data (169 girls and 162 boys, 331 students total) from students who depicted a mathematician as a mathematics teacher in the classroom.

Data Analysis

In this article, instead of seeking the meaning behind each of the drawings, data analysis focused on identifying patterns in the drawings (Haney et al., 2004). The
extent to which ‘classroom instruction’ is more teacher-directed or student-centred is defined as the mode of instruction. The drawings accordingly were analysed through a four-point scale for coding the mode of instruction depicted in student drawings. Table 1 shows these scales and gives a list of indicators illustrating what constitute each of them.

<table>
<thead>
<tr>
<th>Scale</th>
<th>Indicators</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Highly student-centred mode of instruction</td>
<td>Student desks are clustered</td>
</tr>
<tr>
<td></td>
<td>Students are working in groups/pairs</td>
</tr>
<tr>
<td></td>
<td>Teacher talk, if any, invites discussion (e.g., praises, questions)</td>
</tr>
<tr>
<td></td>
<td>Active learning is apparent (students are engaged in an activity)</td>
</tr>
<tr>
<td></td>
<td>Teacher is with/nearby students.</td>
</tr>
<tr>
<td>3-Moderately student-centred mode of instruction</td>
<td>Student desks are usually clustered. If desks are in rows, active learning should be apparent (i.e. students are engaged in an activity)</td>
</tr>
<tr>
<td></td>
<td>Students are seated in groups/pairs</td>
</tr>
<tr>
<td></td>
<td>Teacher is at a distance (at blackboard or at teacher’s desk)</td>
</tr>
<tr>
<td></td>
<td>At least two people (two students or one student-one teacher) are included in the picture and there should be interaction (e.g., content-related talk, engaged in an activity collectively). If only one student is present, active learning should be apparent (i.e. the student should clearly be engaged in an activity)</td>
</tr>
<tr>
<td>2-Moderately teacher-directed mode of instruction</td>
<td>Student desks are in rows.</td>
</tr>
<tr>
<td></td>
<td>Students are seated in rows.</td>
</tr>
<tr>
<td></td>
<td>If depicted, the teacher is at a distance (at blackboard or at teacher’s desk) and lecturing. If the teacher is not depicted, there should be at least one student present in the picture</td>
</tr>
<tr>
<td>1-Highly teacher-directed mode of instruction</td>
<td>Only the teacher depicted, students are not present in the picture.</td>
</tr>
<tr>
<td></td>
<td>If depicted, student desks are in rows.</td>
</tr>
<tr>
<td></td>
<td>The teacher is depicted at the blackboard, or at teacher’s desk.</td>
</tr>
<tr>
<td></td>
<td>Teacher talk, if any, is lecturing or disciplining.</td>
</tr>
</tbody>
</table>

Table 1: Guideline for analysing the mode of instruction in student drawings (Gulek, 1999)

Each scale was unpacked in the form of indicators represented in the drawing and/or writing and coded as ‘3’, ‘2’, or ‘1’. The narrative descriptions of students were used in assisting the coding and allowing to confirm or reconsider the interpretations. When represented, the teaching materials such as whiteboards, books, and concrete materials, and students’ attitudes, feelings, or emotions were also noted. The author and a second researcher in the team independently coded a subsample of the thirty DAMT responses achieving 96% agreement. Disagreements were resolved through discussion to reach consensus. The data was then coded by the author; throughout the analysis, the author consistently attempted to discuss and resolve issues that required further attention for consensus with the second researcher.
In Figure 1 (see Appendix A), typical examples of student drawings and descriptions are given to illustrate Highly teacher-directed (Figure 1a through 1c) and Moderately teacher-directed mode of instructions (Figure 1d through 1f), the two most commonly represented drawings in this study.

Results

The analysis of the grade 6 students’ depictions (Table 2) revealed that more than half of the students pictured a Highly teacher-directed (55.58%) and two-fifth of the students pictured a Moderately teacher-directed mathematics classroom (40.48%).

<table>
<thead>
<tr>
<th>Highly student-centred</th>
<th>Moderately student-centred</th>
<th>Moderately teacher-directed</th>
<th>Highly teacher-directed</th>
<th>Not clear</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 (0.30%)</td>
<td>134 (40.48%)</td>
<td>184 (55.58%)</td>
<td>12 (3.62%)</td>
</tr>
</tbody>
</table>

Table 2: The mode of instruction as depicted in drawings (N = 331)

Students were not represented in many of the Highly teacher-directed depictions (e.g., Figure 1a through 1c), with only sixteen drawings including images of students. When depicted, students sit in a row, one behind the other or side by side. Student comments are contradictory to their drawings. One student indicated that she really grasps what the teacher explains to them, and another student wrote that he does well in the mathematics exam and wishes that he would always perform well. However, fourteen other students’ attitudes that were reflected in their drawings were more negative. On four drawings, students were disengaged; they were pictured as either misbehaving or listening to music and playing computer games that were not connected to classroom activity. In ten drawings, students were depicted as unhappy because the teacher annoys them.

In these drawings, the teacher was mostly pictured at the whiteboard (161 depictions) or at the teacher’s desk (23 depictions) when lecturing, demonstrating, explaining, or disciplining. Both in the drawings and written descriptions, there were strong indications of two roles of the teacher: lecturing and disciplining. Students’ descriptions included: “Who will make this calculation?” “Hi, class! We will study Integers today.” “You girl! Why didn’t you do your homework? Copy the board down into your notebook, come on!”

In Moderately teacher-directed drawings (e.g., Figure 1d through 1f), images of students were present. In 91 of these drawings, they sit in a row of one or two students, one behind each other or side by side. In others, students were generally portrayed at the board solving mathematics questions. Within this group, in most drawings (n = 85), there were no hints of the students’ attitudes or feelings. Among the remaining drawings, students were portrayed as smiley or happy (n = 30). In some drawings, students were depicted as engaged in learning (18 mentions), wherein they raised their hands to volunteer to solve the question on
the whiteboard or listen to the teacher attentively. In only one drawing, students were depicted as disengaged. The creator of this drawing wrote that students were misbehaving. In five drawings, students were pictured as unhappy. As reported by the creators, this unhappiness stemmed from the fact that the depicted students could not solve the question or could not comprehend mathematics. In one drawing, it was written: $1/2 + 3/4; 2/5 + 4/8$ on the student’s paper who looked unhappy because she could not solve it. In another one, the student wrote: “Whoever knows the question gives the answer; others hide under the table.”

The teacher was often depicted at the whiteboard at a certain distance from students (116 depictions) or sitting at the teacher’s desk (18 depictions). Quotations such as: “Kids, look here [whiteboard]!” , “Solve this question.” or “Let’s solve this equation.” were indicative of the teacher’s main activity: lecturing. No evidence of group work or any other student-centred methods existed. Nor was there any indication of activity apart from solving mathematics questions, or interaction among students, or between students and the teacher beyond practising mathematics questions such as: “$2/3 – 6/3 = ?$, $7/8 + 8/8 = ?$” or “$14 + 13 = 27; 10 + 10 = 20; 19 – 10 = 9$”. In fact, the mathematical content of most of the drawings in the whole sample was basic arithmetic.

Sadly, within the whole sample, none of the drawings showed a Highly student-centred classroom. Only in one drawing (Figure 2a) there was slight indications that might suggest a Moderately student-centred mode of instruction. In this drawing, two students were depicted in front of the board, discussing the solution for a question described by the creator.

Our teacher seated at the desk solves math questions. The students comment on the teacher’s solution. One of them says not that but this, the other one next to him explains that it is wrong. (a)

There is our math teacher, Ceren [a friend] and I. (b)

There are the teacher and the student. (c)

Figure 2: Examples of drawings might suggest Moderately student-centred (a); Moderately teacher-directed or Moderately student-centred instruction (b and c)

The remaining twelve drawings are not included in the four-point scale (3.62%), as it was difficult to decide whether the mode of instruction depicted was Moderately teacher-directed or Moderately student-centred (see Figure 2b and 2c). These drawings picture the teacher standing at the whiteboard (9 depictions) or at teacher’s desk (3 depictions). However, in all of them, students were next to
the teacher, and sometimes with smiley faces. This possibly indicated that they were quite happy doing mathematics, but there is no hint as to whether there was a content-related discussion between the teacher and students based on a particular task or a student-centred task was in use.

In the depictions, a whiteboard (301 mentions) or books (19 mentions) were the most remarkable teaching resources. Only in one depiction a smart board and in eight depictions concrete materials such as a ruler (7 mentions), geometric objects (2 mentions), a compass, a protractor, or a miter (3 mentions) were present. In ten depictions there was no indication as to the teaching resources that might have been used. Different materials and tools such as calculators, computers, or technological or digital tools did not appear in any drawings.

The resources shown on the drawings did not vary according to the mode of instruction, but students’ attitudes or feelings did. In drawings where the mode of instruction was identified as Moderately teacher-directed or potentially Moderately student-centred, students described themselves as being happier than in drawings where the mode of instruction was Highly teacher-centered.

Concluding comments

The analysis of student drawings and writing revealed that 96% of students depicted either Moderately or Highly teacher-directed mathematics classrooms, mostly a whiteboard and/or books represented in drawings as the teaching and learning resources, and there were strong references to the computational manipulations. Students’ (DAMT) drawings in the previous study fell into two distinct groups with the present article only providing the data regarding the drawings that clearly represented a mathematics teacher in the classroom. The results are therefore being regarded in isolation from the remaining data from the overall study and should be interpreted with caution. Also, the sample might not be representative of the entire population of six grade students within Turkey or in other countries. Nevertheless, the study contains three implications and directions for future research.

First, my observation is that students’ depictions might mirror their classroom experiences. Most students pictured their actual mathematics teachers and classrooms, and some expressed their teaching and learning practices. The classroom environment that emerged from these depictions are worrying because these teacher-centred approaches negatively impact students’ attitudes (Hasni and Potvin, 2015), seeing mathematics mostly as “numbers” or “lots of formulas” (Boaler, 2015) and making it difficult for students to remain engaged in mathematics (European Commission [EC], 2011). Such trends have longer term implications for students’ mathematics learning and it is my recommendation that this area be a focus of further research.

As Losh et al. (2008) found, students in this study took the drawing task seriously and put some considerable amount of effort and thought into completing it. The
student drawings in this study “can provide a valuable catalyst to document, change, and improve what goes on in [Turkish] schools.” (Haney et al., 2004, p. 243). Second, I suggest that teachers can use drawings to access and become aware of student views about mathematics teaching and learning and use such understanding as a basis for reflecting on their own practices. Student drawings might also inform policy makers about the impact of curriculum revisions on classroom teaching.

Results from TIMSS studies show that Turkish lower secondary students have certain shortcomings in mathematics achievement, being far below the OECD average (Mullis, Martin, Foy and Hooper, 2015). Finally, I believe future studies on how students’ classroom experiences correlate with their performance would contribute our understanding on the possible factors behind student low performance in mathematics. TIMSS studies also show that between-school variation in Turkey is quite large and explain more than 60% of differences in student achievement (EC, 2011). The previous study, part of which is presented here, was implemented in twenty metropolitan schools. Students’ experiences of classroom mathematics in disadvantaged or remote schools would extend the findings of this study.

Acknowledgement: I wish to acknowledge the students and teachers who participated in, the second researchers A/Prof Bulent Cetinkaya who showed commitment to, and the Directorate of National Education which supported implementation of the study.

References


Appendix A: Typical examples of student DAMT responses

In the drawing I made, a math teacher teaches Polygons to students. (a)

There is a math teacher in that picture writes problems at the whiteboard, and asking them to the students says [the answer is] correct or wrong. (b)

To be identified, I wrote the words math teacher uses in disciplining students. [on the picture: e.g., Shut up!] (c)

Kalim Hoca [their teacher] [a pseudonym name] writes calculations at the whiteboard. The students write these calculations on their notebooks. (d)

[Does anyone who not understand?] The ones sit on desks are us [students]. The one at the whiteboard teaching is our mathematics teacher. (e)

There is a math teacher in the picture and this teacher teaches calculations to the students. (f)

Figure 1: Examples of drawings and descriptions illustrating Highly teacher-directed (a through c) and Moderately teacher-directed (d through f) mode of instructions

LEARNING TO ENHANCE EMERGENT BILINGUALS’ ACCESS TO MATHEMATICS: ELEMENTARY TEACHERS EXPERIMENTING WITH THE CLINICAL INTERVIEW

Hanna Haydar

Abstract

We report in this paper on a study of prospective and beginning teachers learning to conduct clinical interviews to enhance emergent bilingual students’ access to mathematics. The study is conducted within the context of a teacher education program in New York City. The student populations in the U.S. schools continue to become more
diverse, with the emergent bilingual population representing the fastest growing group. Expanded learning opportunities and instructional accommodations should be available to emergent bilinguals who need them to develop mathematical understanding and proficiency. We will report on how teachers develop and/or modify interview protocols to support their teaching of mathematics to children whose first language is not English. A meta-analysis of various interviews will be used to emphasize findings on how this experience affects beginning teachers’ lesson planning skills, and helps them understand and develop strategies to teach mathematics to their emergent bilinguals. We will zoom in on a case study of an elementary teacher who took these methods further and developed a structure for interviewing as a pre-lesson-planning assessment and for incorporating peer interviewing.

Keywords: clinical interview, mathematical thinking, teacher education, formative assessment, emergent bilingual, English language learners

Introduction

This paper describes how elementary mathematics teachers use the Clinical Interview method to make mathematics more accessible when working with emergent bilingual students. We report on a study of prospective and beginning teachers learning to conduct such an interview within the context of teacher education program in New York City. It looks more closely on one case study of a 3rd grade teacher who took the interviewing further by conducting an action research to improve her lesson planning and differentiation practices.

Emergent Bilinguals and Access To Mathematics

The student populations in the U.S. schools continue to become more diverse, with the emergent bilingual population representing the fastest growing group. There is broad consensus that more needs to be done to develop equitable and quality mathematics instruction to meet the learning needs of emergent bilinguals (Foote, 2010) to narrow the persistent mathematics achievement gap. Brooks (2016) argues that when teachers view students through the lens of a policy classification such as a label, they miss vital opportunities to develop instruction tied to their diverse linguistic experiences (Brooks, 2016). Ascenzi-Moreno (2017) emphasizes the powerful role that teachers hold in negotiating and shaping policies related to language use in their classroom. She advocates that teachers should be encouraged to inquire into who their students are and to understand what resources they bring into the classroom in order to forge innovative ways of working with them.

According to the National Council of Teachers of Mathematics (2013) “expanded learning opportunities and instructional accommodations should be available to English language learners (ELLs) who need them to develop mathematical understanding and proficiency.” Language Educators call teachers to consider both their emergent bilinguals’ receptive and productive languages. Receptive language points to how students process language to comprehend information,
ideas or concepts. Productive language deals with how they use language to express information, ideas, or concepts. In this study, we adopt the Framework for Creating Access (Driscoll, Nikula and DePiper, 2016) that advocates: (a) Challenging mathematical tasks (b) Multimodal representation (c) Development of mathematical communication and (d) Repeated structured practice.

In addition to a focus on visual representations, two ELA strategies were selected for the initial stage of this study, participants were trained on the use of “Sentence starters and frames” that targets students’ productive language and “Three reads” that helps with text and word problem comprehension:

**Sentence starters and frames**

Sentence starters and frames help provide opportunities to scaffold ELLs productive language. They can be used to support student explanation, scaffold group or class dialogue, and be used to engage with mathematics vocabulary and math-specific word meanings in context. (Baffington, Knight and Tierney-Fife, 2017).

**Three reads strategy**

Three Reads strategy uses three readings of the text in order to make sense of the word problem or task: the first read is to get a sense of the context in order to understand the “story” or big idea of the text; the second read is to discern the question or purpose of the text. The problem is read again in its entirety, looking specifically for information about what needs to be answered or done to be successful; and the third read of the text is to gather important information that is needed to solve the problem or achieve the purpose of the task, such as specific quantities and their relationships.

**The Clinical Interview**

Our research adopts the various calls for the use of clinical interview by teachers in their classroom instruction, as a powerful and necessary mean to help them understand their students’ mathematical thinking (NCTM, 2000; Haydar, 2003; Hunting, 1997; Long and Ben-Hur, 1991) The strength of the method created by Jean Piaget (1952) was in the creation of a method that depicts the child’s “natural mental inclination”, identify underlying thought processes and account for the larger “mental context” (Ginsburg, 1997).

**Elementary Teachers learning to interview**

Participants in this study are pre-service and in-service teachers registered in a graduate program in childhood mathematics education. In a sequence of four graduate mathematics education methods and research courses teachers are trained to use the clinical interview with their own students and progress in their experience from learning about the method to conducting interviews with their students about number sense, algebraic and geometric reasoning. They focus
within the third course on how to use the technique to make mathematics more accessible when working with emergent bilingual students and finally they use in their final course a set of interviews in the context of action research (Haydar, 2017).

**The Interview Protocol**

Participants used the following protocol to select their interview tasks and conduct clinical interviews centered on the use of the above ELA’s strategies. The protocol provided a structure for interviewers to emphasize visual mathematical representations and consider both receptive and productive languages of the interviewee. The protocol left however the choice of tasks and the selection of specific visual models and other ELA strategies up to the teachers to promote a sense of ownership and maximize the pedagogical rehearsal:

| 1. Display a word problem mathematics task |
| 2. Use appropriate language access strategies (e.g. Three Reads) |
| 3. Give student time to work individually on creating a visual representation for the task |
| 4. Consider how to support language production based on the created visual representation (e.g. Sentence Starters and Frame…) |
| 5. Display another mathematics task |
| 6. Use appropriate language access strategies (e.g. Three Reads) |
| 7. Give student time to work individually to complete sentence starter “I wonder…” about the sample of another student’s work. |
| 8. Ask student to make sense of each step of the sample student’s approach |
| 9. Have a conversation with your subject about the other student’s use of visual representation |

**Methodology**

Videos of 39 participants’ interviews were viewed by two researchers to determine critical events (Powell, Francisco and Maher, 2003) revealing how teacher participants were learning how to enhance emergent bilinguals’ access to mathematics when attempting to solve complex word problems. These events were then transcribed. Interview transcripts and participants’ reflection journals were then coded. In line with methods from grounded theory (Charmaz, 2006), the transcripts and journal data were sorted, resorted and analyzed, moving from descriptive information to constructing explanatory schemes (Corbin and Strauss, 2014). As a follow-up, and to examine how teachers adapt and experiment with the clinical interview further in their work with emergent bilingual students we studied the case of Lisa, a third grade teacher. Case studies focus on a single
unique entity in the purpose to make visible “what some phenomenon means as it is socially enacted within a particular case” (Dyson and Genishi 2005). The teacher’s action research report, reflection journals, videos of 5 of her math lessons about fractions and 6 pre and post-clinical interviews were analyzed using the same methods described above.

**What teachers learned**

Findings are revealing common themes when elementary mathematics teachers experiment with clinical interviewing to try out ELA techniques and make mathematics more accessible for Emergent Bilinguals:

1. Considering the Child and language demands in advance: *Planning for the clinical interview created a space for teachers to consider the child’s mathematical and language demands a priori.*

2. Interview moments with classroom implications: *The clinical interview consisted a safe space for teachers and emergent bilingual students to try out, evaluate and decide on what worked and needed to be implemented in the classroom.*

3. Using the visual representations for eliciting problem solutions: *The clinical interview consisted a safe space for teachers to push students to draw diagrams or visual models. We observed also how many teachers used this visual representation to scaffold the emergent bilingual child’s problem solving and bridge between his/her verbal presentation and mathematical answer.*

**Taking The Interview To The Classroom: Action Research**

Action research provides for teachers “a framework that guides [their] energies toward a better understanding of why, when, and how students become better learners” (Miller, 2007). Most teachers use clinical interviewing as part of their data tools to help them answer their research questions. Few teachers however, decide to take the clinical interviewing one step further and choose it as the teaching practice to further experiment with and investigate.

To illustrate how the action research becomes a space to improve teachers’ interviewing skills and allow them to experiment with it in their regular classroom we will present here the case of a third-grade teacher, who experimented further with clinical interviewing in attempt to maximize the access of her emergent bilingual students to mathematics:

**What Lisa did: More interviewing for better inclusion of Emergent Bilingual voices in the lesson planning**

Lisa was a 3rd grade general education classroom teacher at a public school in Brooklyn, New York at the time of the study. Lisa’s class consisted of 27 students where 18 students were currently classified as ELLs/former ELLs and 4 students
had an IEP for related services such as counseling, speech, and occupational therapy. For her action research, Lisa decided to explore how she could use the clinical interview method to improve her instruction of fractions to 3rd graders. Based on her interviewing experience described above she planned to go further and experiment with two aspects: 1) Interviewing as a pre-lesson-planning assessment and 2) peer interviewing.

**Interviewing as a pre-lesson-planning assessment**

Lisa planned to use the clinical interview as pre-assessment method to better understand the strengths and challenges of her emergent bilingual students while they develop fraction understanding. She intended to use the findings of the interviews as insights for targeted lesson planning:

“I believe that the clinical interview method would be a good tool to use for researching fraction development and understanding because it could give me insight to where struggles lie and what misconceptions arise. With that knowledge, I could then plan instruction accordingly to target those issues in learning and reassess throughout my teaching and at the conclusion of my teaching with another interview.” (Lisa, 3rd Grade Teacher)

**Interviewing Nathalie**

Nathalie is an emergent bilingual student who always tries her best, is willing to take risks and participates in classroom activities. Nathalie processes information very slowly and is very slow to complete tasks. Lisa believed that paper assessments do not accurately portray Nathalie’s mathematical understandings because aside from computation assessments, they require students to do a lot of reading and writing and she performs below grade in both of those content areas.

“I chose to interview Nathalie, because although I consider her to be a struggling math student based off of her assessments she often surprises me with the mathematical understandings she can verbalize. I wanted to interview Nathalie so that she would have ample opportunities to showcase her understanding of fractions.”

After this clinical interview, Lisa concluded that Nathalie had already developed the big ideas that pieces don’t have to be congruent to be equivalent and fractions are part/whole relations and the misconception that the greater the denominator the bigger the fraction. In addition, Nathalie was starting to develop an understanding of the following big ideas: If numerators are common only denominators matter when comparing and If denominators are common only numerators matter when comparing

The interview gave Lisa a better insight on Nathalie’s struggles with her productive English language:

In this interview, I found that Nathalie struggles dearly with explaining her mathematical thinking. Even with the support of a visual representation Nathalie seems to be unable to find the words to explain her ideas. Based off of her answers, I believe that she has started to develop some key understandings of fractions, but
the explanations she was giving were unclear and one could tell she was beginning to feel frustrated with her inability to explain her thinking. For example, when asked to complete the number sentence $\frac{3}{6} \text{____} \frac{3}{8}$ Nathalie selected the greater than sign.

Teacher: “How do you know?”

Nathalie: “Because um, if I put this right here then it will be like…[unclear speech]…because if this one was that means that it means like I will put less than and 8 is less than and 6 is greater than. Because it’s almost like um…it’s almost like um…”

Teacher: “Do you think you need to represent it with pictures?”

After some scaffolding

Nathalie: [draws Figure 1]

![Figure 1: Nathalie’s drawing](image)

Teacher: “Why is three sixths greater than three-eighths? Maybe you could use your picture to help you explain.”

Nathalie: “Because in this one it’s um this one this one is like sixths. 3 are the same. And 6 and 8 are not the same.”

Teacher: “They’re not the same. You’re right. So in three-eighths and three-sixths, which part of the fraction is the same? What do you call that?”

Nathalie: “The numerator.”

Teacher: “The numerator. So you are saying that 6 and 8 are not the same, they are different. So our denominators are different. So that’s why three-sixths is greater than three-eighths?”

It is evident that Nathalie was beginning to develop some big ideas surrounding fractions but struggling to verbalize those understandings. But she knew she should be paying attention to the denominator because the fractions had common numerators. However, it was unclear if Nathalie understood that there is a relationship between the number in the denominator and the size of the piece.

**Implications for teaching:**

Based on the results of her interviews with Nathalie and two other emergent bilingual students Lisa made decisions related to both the mathematical content
and facilitation of math talk for her emergent bilingual students. “I realize that during my teaching experiment I must provide my students with opportunities to explore fraction comparisons with common numerators and denominators and to talk about these comparisons. I plan to put students in strategic partnerships, in hopes that students will question one another or even argue about their disagreements to help develop their reasoning skills.” She noticed how evident that most students are unable to provide a sound mathematical explanation and that she needed to highlight the usefulness of visual models such as bar models or number lines when comparing fractions, “while analyzing my students’ assessment and explanations I found that very few of them used them as a thinking tool”. We will describe in the following section how Lisa incorporated peer interviewing as an attempt to create more opportunities in her lessons that facilitate students’ productive language while explaining their mathematical work.

**Peer Interviewing**

Ginsburg, Jacobs and Lopez (1998) called for the use of peer interviewing. This involves teaching children to interview one another. In peer interviews, students learn about each other’s thinking, they learn about the mathematics content of the interview and about each other’s feelings. When working with emergent bilingual students, one additional strength of peer interviews is that they make “children’s thinking much more overt” by bringing to light events that otherwise remain hidden. The Peer Interview “brings more of the cognitive processes out into the open where teachers and students can examine and try to understand them” (Ginsburg, Jacobs and Lopez, 1998). Lisa chose to plan a whole lesson of her teaching experiment using peer interviewing, she prepared task cards incorporating sentence frames to facilitate the participation of emergent bilingual students:

“Peer interviews will give students another opportunity to practice explaining their thinking, but it will also strengthen their listening skills. I plan to give each student a task card, which will either have a number sentence that they must determine if it’s true or false or an incomplete number sentence that students must complete with comparison notation. One partner [partner A] will read their card and answer it. Partner A must explain their thinking to partner B who must ask questions to clarify when necessary and then will record 2-3 things about partner A’s response. Then partner A and B will trade roles.” (Lisa, 3rd grade teacher)

Peer interviewing puts the child in a position of responsibility for listening to a partner, asking questions and then reporting on their discussion. The way Lisa designed the activity and required partners to record their conversation held them more accountable for their work.

The lesson with peer interviews was the only lesson where all students including emergent bilingual students asked more questions than the teacher. Lisa realized that when she took a step back because of the nature of the lesson, students took
more ownership in their learning and were willing to ask one another questions. One main positive finding from analyzing the critical moments of the lesson was that there were many moments in which students explained their mathematical thinking. This results from Lisa’s conscious decision to maximize in her lesson plans “ample opportunities to explain their mathematical thinking in either a verbal or written format”. After analyzing all her data, Lisa reflected that although her teaching was effective it was not long enough. She realized that her students needed more time to work with the complex fraction big ideas and planned on using more opportunities for her emergent bilingual students to practice explaining their thinking and reasoning in order to produce strong mathematical arguments.

Conclusion

This study is showing how interviews can become safe spaces for teachers to understand and rehearse tools, methods and approaches for making mathematics more accessible for their emergent bilingual students. Findings of this study intersect with similar results from previous work that showed the transformative experience that can be provided by the clinical interview. The themes revealed in this study: consideration of the language demands, centering classroom instruction around student thinking and focusing on visual representations are all important practices included in most of the current international curriculum standards. The added element here is the use of the method to work on the skills of noticing and enhancing the mathematics of emergent bilingual students. These findings echo some of the principles deriving from teacher’s “noticing” (Sherin, Jacobs and Phillip, 2011) and “rehearsal” (Ghousseini, 2017) constructs. Subsequent studies that explore these relations might help us better interpret the place and effects of the use of clinical interviewing by teachers in linguistically diverse classrooms.

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**PRIMARY SCHOOL STUDENTS LENGTH ESTIMATION COMPETENCE - A CROSS-COUNTRY COMPARISON BETWEEN TAIWAN AND GERMANY**

*Jessica Hoth*, Aiso Heinze, Dana Farina Weiher, Silke Ruwisch and Hsin-Mei E. Huang

**Abstract**

Estimation competence is assumed to be a relevant skill that needs to be addressed in the mathematics classroom (Sowder, 1992; Bright, 1976). However, little is known about cross-country differences in primary school students’ length estimation competence and what impact different opportunities to learn in the different educational contexts have on this competence. Taking into account an extensive analysis of

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estimation situations, Heinze et al. (2018) proposed a model that holds 72 estimation situations as a basis for assessing length estimation competence. Based on this model, a test on length estimation competence was developed and used in 43 German and Taiwanese classes. In total, 903 third and fourth grade students fulfilled the length estimation tasks. The test on students’ length estimation competence proved to be partially measurement invariant. Cross-country comparison revealed that Taiwanese students outperform the German students in length estimation.

**Keywords**: length estimation competence, length estimation situations, opportunities to learn, cross-country comparison, primary school

**Introduction**

Estimation competences are frequently needed in everyday situations such as driving through speed limits that only apply to a prescribed length or estimating how many kilometers you’ll drive with your car in a year in order to get the right insurance class. Therefore, estimation competences are central mathematical skills that are considered in curricular guidelines in several countries (e.g. NCTM, 2000; KMK, 2004; Ministry of Education, 2010).

However, there is still a lack of research on students’ estimation competence as well as a lack on models about the estimation processes (Joram et al., 1998). In particular, it is still unclear, whether the development of students’ length estimation performance depends on the cultural context in which students are grown up and in which they are educated. Following the idea that students’ estimation competence is based on specific knowledge (Sowder, 1992) and skills (Joram et al., 1998; Siegel et al., 1982), students from different schooling contexts which have experienced different opportunities to learn should differ in their estimation competence. Since most empirical findings on length estimation competence stem from studies from one country, there are hardly cross-cultural comparisons. Therefore, the research aim of the following paper is to analyze primary school students’ length estimation competence in a cross-country comparison between Taiwan and Germany. These two countries have very different teaching traditions with different opportunities to learn and both use teaching languages that different from English which may result in a very different length estimation competence.

**Theoretical background**

A general definition of estimation is provided by Mitchell et al. (1999, p. 9) who define “Estimation [a]s a process whereby one approximates, through rough calculations, the worth, size, or amount of an object or quantity that is present in a given situation. The approximation, or estimate, is a value that is deemed close enough to the exact value or measurement to answer the question being posed”. As implied by this definition, there are different estimation contexts or estimation tasks that can be addressed. It is generally differentiated between (1) estimation results of computations, (2) estimating measures and (3) estimating numerosity.
(Sowder, 1992; Hogan and Brezinski, 2003). In our research, we concentrate on measurement estimation.

**Measurement estimation**

For measurement estimation, Bright (ibid., p. 89) defines that “Estimating is the process of arriving at a measurement or a measure without the aid of measuring tools. It is a mental process though there are often visual or manipulative aspects to it”. Children need specific knowledge in order to fulfill measurement estimation tasks (Hiebert, 1981; Joram, 1998). In particular, knowledge about physical measurement is assumed to be a prerequisite for measurement estimation (Joram et al., 1998). Another knowledge aspect that is mentioned by Sowder (1992, p. 383) is knowledge about types of measures or basic conceptual knowledge that can be used as a reference to estimate. If a given picture shows a building whose height should be estimated with a man of average height next to it, students must know how tall a man is generally in order to use this knowledge as a reference for their estimates. Therefore, it can be assumed that children from different schooling contexts with different opportunities to learn and different knowledge bases might vary in their measurement estimation competence.

Interestingly, empirical studies focusing on measurement estimation are based on items addressing very diverse estimation situations though they all assume to assess (the same?) measurement estimation competence (e.g. Swan and Jones, 1980; Siegel et al.; 1982; Hogan and Brenzinski, 2003). Estimation situations vary for example in regard of the measures in focus, the accessibility of the to-be-estimated object (TBEO), benchmark or the size of the objects. To date, a systematical approach to use different estimation situations in the course of competence assessment is missing. Against this background, the present study focuses on one specific facet of measurement estimation (length estimation) and uses a systematic approach of relevant aspects regarding length estimation situations in order to assess length estimation competence.

**Measurement estimation situations as a basis for assessing measurement estimation competence**

As Bright (1976) points out, there are different – mainly eight – estimation situations that must be addressed in the mathematics classroom in order to foster students’ length estimation competence. He distinguishes whether an object is given and one specific characteristic of this object should be estimated or whether a measure is given and students should estimate on objects that complies with the given measure. In each case, the TBEO can be physically present or not and the unit is present or not (ibid.).

Starting from this model by Bright, Heinze et al. (2018) identified seven characteristics of estimation situations that are mostly pairwise independent:

1. **Estimation condition**: TBEOs can be physically present or not,
2. **Accessibility**: TBEOs can be touchable or not,
(3) **Activity**: situations can either require to give measure as an estimate or might require to construct a representation of a given length (e.g., by drawing a line),

(4) **Role of the Benchmark**: in a situation no benchmark may be given or a specific benchmark is mentioned but not visible, is visible in real size (or not in real size), or is even visible in real size and touchable,

(5) **Length of the Benchmark**: If a benchmark is given, its length may be provided explicitly or not,

(6) **Scale**: situations ask for estimates in standard units (e.g. metric units like cm) or in non-standard units (e.g., room width in number of floor tiles),

(7) **Size**: TBEOs can vary in their length from small to large.

Combining the different features of these six characteristics, Henize et al. (2018) concluded that there are 72 estimation situations. Some combinations do not make sense in real life or in a test situation and were excluded (ibid.).

Heinze et al. (2018) argued that it is relevant for research in the field of measurement estimation competence to consider these different estimation situations because low variation of estimation situations might affect the validity of the empirical data and the interpretations of the results.

Therefore, this model of different types of length estimation situations built the basis for developing a test to measure students’ length estimation competence as described in section 3. With this reference, a systematic and complete theoretical foundation is laid for validly measuring students’ length estimation competence.

**International differences in students’ length estimation competence**

It is known from research that measurement estimation skills are learnable and can be improved by interventions (e.g. Jones et al., 2009; Hildreth, 1983). In addition, it is assumed that measurement estimation competence is dependent on specific knowledge and skills such as knowledge about measurement concepts (e.g. Joram et al., 1998) or spatial abilities (e.g. Sowders, 1992). This knowledge is to a large extent acquired and fostered in the mathematics classroom. Therefore, different schooling contexts such as curricula, teacher education and the education tradition (Blömeke et al., 2014) may have great influence on students’ length estimation competence.

International comparative studies on students’ mathematical abilities show differences in German and Taiwanese students’ mathematical skills (e.g. OECD, 2014). However, due to a lack of international comparative studies on students’ measurement estimation competence, little is known about the differences between German and Taiwanese students’ length estimation competence. Taking in consideration the very diverse educational background, the two different teaching languages than English and the Taiwanese students’ lead in mathematics competence, it can be hypothesized that their length estimation competence and
strategies may differ. In order to identify factors that may influence differences in German and Taiwanese students’ competence, we aim at selecting a specific content area – such as length estimation – and analyze how differences in students’ competence can be traced back to crucial elements in the students’ schooling contexts. In addition, preliminary analyses of the two curricula indicate that length estimation is addressed very differently in the Taiwanese and German classrooms. In Taiwan, there are specific learning opportunities which address length estimation (Huang, 2016) whereas in Germany, length estimation is often “only” an intermediate step in the introduction of length measurement (Franke and Ruwisch, 2010).

Against this background, cross-national comparison of students’ length estimation competence would give insight into the effects of different implementation potentials of this specific mathematical content.

**Research intention and research questions**

Based on the theoretically proposed estimation situations (see section 2.2) which allows a valid assessment of length estimation competence, we consider the question how length estimation competence differs between children from two countries with striking differences in their educational context. Hence, the research question for the following analyses are:

1. Is it possible to reliably assess primary school students’ length estimation competence in Taiwan and Germany with the same test?

2. Are there cross-national differences between German and Taiwanese primary school students’ length estimation competences?

In order to assess students’ competence in length estimation, a length estimation assessment was constructed and evaluated (Weiher et al., in preparation). The test aimed at assessing the length estimation competence of third and fourth grade students. Concerning to Joram et al. (1998) most children are able to learn physical measurement concepts in first and second grade but these concepts seem to be challenging for many students throughout their school career. However, in third and fourth grade, measurement concepts should be initiated and length estimation competence should be measurable.

**Study design and methodological approach**

The previously described theoretical elements of estimation situations built the basis for item development. Length estimation items were constructed that varied on the different characteristics of estimation situations. These items were evaluated and tested in a pilot study with 248 third and fourth grade students from Germany and Taiwan. Some of the items were revised for the main study and, finally, 34 items were used in the main testing (for a detailed description of the test development see Weiher et al., in preparation).
**Test design and test administration**

Each of the 34 items constructed for the formal length estimation assessment can be classified with regard to the previously described theoretical estimation situations. Following the theoretically proposed structure, items in the formal estimation test required the students to estimate measures of TBEOS that were either physically present or not (Estimation condition) and touchable or not (Accessibility). The measures that the students were asked to estimate were either metric or non-metric (Scale) and some questions asked the students to estimate a measure while other questions asked the students to draw a line in the length of a given measure (Activity). In some items, a benchmark was prescribed that the students could use for their length estimations while other items did not prescribe benchmark (Role of Benchmark). For items that held prescribed benchmarks, it was varied whether the length of the benchmark was given or not (Length of Benchmark). Finally, the TBEOS varied in size. Some items introduced TBEOS that were small (TBEO \( \leq 12 \) cm) other items proposed TBEOS that were not small (> 12 cm).

Two item examples are shown in Figure 1.

<table>
<thead>
<tr>
<th>Item example 1</th>
<th>Item example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>How long is the yellow ribbon at the blackboard?</td>
<td>Draw a straight line, which is about three times as long as this stripe.</td>
</tr>
<tr>
<td>The yellow ribbon is about _______ cm long.</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 1:** Item examples of the formal estimation test

In the current study, the 34 length estimation items were presented as a test booklet and provided for each students who participated in the study. The participating classes were tested in their individual classrooms. The testing time was one class period. Student questions that occurred in the individual working time, were answered using an answering manual that was also developed during the pilot study. If some of the students did not complete their work after one class period, they had to stop and hand in their booklet.

**Sample size and characteristics**

The sample size and its characteristics are shown in Table 1. In total, the sample spreads evenly between countries. In addition, the amount of participating third and fourth grade students as well as male and female students is almost equal in both countries.
### Table 1: Sample Size of the formal estimation test

<table>
<thead>
<tr>
<th></th>
<th>Number of students</th>
<th>Number of female students (percentage)</th>
<th>Number of classes</th>
<th>Number of third grade students (percentage)</th>
<th>Number of third grade classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>903</td>
<td>474 (52%)</td>
<td>43</td>
<td>442 (49%)</td>
<td>22</td>
</tr>
<tr>
<td>Taiwan</td>
<td>453</td>
<td>232 (51%)</td>
<td>18</td>
<td>220 (49%)</td>
<td>9</td>
</tr>
<tr>
<td>Germany</td>
<td>450</td>
<td>242 (54%)</td>
<td>25</td>
<td>222 (49%)</td>
<td>13</td>
</tr>
</tbody>
</table>

**Data analyses**

The students’ answers to the test items were coded with regard to their deviation from the original lengths of the TBEOs. If the length estimation did not deviate more than 10% from the original length, the answer was coded with 3 points. If the deviation was greater than 10% but not more than 25%, the answers were coded with 2 points. Length estimations that deviated more than 25% but less than 50% got 1 point and, finally, answers that deviated more than 50% were coded with 0 points. Since the time limitation of one class period offered the students enough time to finish the 34 length estimation items, the lesson’s end functioned as a time limitation. Therefore, missing responses were considered as a lack of competence and coded with 0 points.

The data of the test was scaled using a partial-credit model of item-response-theory (Masters, 1982) and the software Conquest was used for data analysis (Wu, Adams and Wilson 2007). In this process, individual ability values are estimated (WLE, Weighted-Maximum-Likelihood-Estimates, Warm, 1989) parallel to item difficulties. The overall test showed an acceptable WLE-reliability of .81. There were no floor or ceiling effects. Three items were excluded from the analysis due to a low discrimination. The discrimination for the remaining items was greater than .23 for all items.

To examine research question 1, multiple-group IRT analyses were conducted to analyze whether the test shows measurement invariant across countries (Taiwan and Germany) (Bock and Zimowski, 1997). More precisely, differential item functioning (DIF) was analyzed in order to test measurement invariance across the two country subgroups. In this analysis, item difficulties are allowed to vary across the two groups and it was analyzed whether item difficulties differed between the two countries. Referring to the classification criteria by Pohl and Carstensen (2012), we identified items with strong DIF (absolute difference in estimated difficulty greater than 1) and let these items be estimated freely across
groups. In addition, we compared a model that allowed for item by group interactions to a model that only allows for main effects of the grouping variable (similar to the proceedings of Pohl and Carstensen, ibid.).

In order to address research question 2, the individual ability values (WLEs) were used for further analysis. In particular, means of groups were compared using t-tests (in order to identify cross-national differences in children’s length estimation competences.

Results

The analysis showed that five items had a strong DIF. In three of these items, the item difficulty was much higher for Taiwanese students while the other items showed higher difficulties for the German students. These five items were estimated freely across the two groups in the subsequent analyses. However, only 16% of the items were freed in the measurement model which lies below the cut-off criteria that Dimitrov (2010) suggested (not more than 20%) and the test can be classified as partially invariant (Brown, 2006).

In addition, we compared the model that only included the main effect of both countries against the model that additionally modeled interaction effects of item and group. Table 2 shows the information for both models. BIC (Bayesian information criterion, Schwarz, 1978) was used as an information criterion to compare the models. Lower values indicate better fitting models while Raftery (1995) already suggests differences of more than 10 as a strong evidence for a better model fit. Here, the BIC of the model involving only the main effect is 39 lower than the model including both, main effect and DIF. This, again, indicates the measurement invariance.

<table>
<thead>
<tr>
<th>DIF variable</th>
<th>Model</th>
<th>Deviance</th>
<th>No of parameters</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>country</td>
<td>Main effect</td>
<td>56292</td>
<td>80</td>
<td>56838</td>
</tr>
<tr>
<td></td>
<td>DIF</td>
<td>56146</td>
<td>106</td>
<td>56977</td>
</tr>
</tbody>
</table>

Table 2: Model comparison with and without DIF

The results of the cross-national comparison indicate that there is a significant difference between German and Taiwanese students’ competence to estimate the length of given TBEOs (see Table 3). The table shows the mean of the Taiwanese and German students’ individual ability values (WLEs). We transformed the mean for the entire sample to 50 and the standard deviation to 10. It turns out, that Taiwanese students outperform the German students.
Table 3: Means (SD) of cross-national comparison on length estimation competence

<table>
<thead>
<tr>
<th></th>
<th>$n$</th>
<th>$M$</th>
<th>$SD$</th>
<th>$t$-test results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taiwan</td>
<td>453</td>
<td>51.47</td>
<td>8.16</td>
<td>$t(814) = 4.478$, $p &lt; .001$,</td>
</tr>
<tr>
<td>Germany</td>
<td>450</td>
<td>48.52</td>
<td>11.38</td>
<td>$d = 0.298$</td>
</tr>
</tbody>
</table>

Exploratory analyses suggest that this effect is mainly based on estimation situations with small and touchable TBEÖs. For test items based on estimation situations with TBEÖs that are not small and not touchable, this difference disappears.

Conclusions

Children in Taiwan and Germany are educated in very different educational traditions. Despite these differences we were successful in the development of a length estimation test that (1) is based on different types of estimation situations to ensure validity and that (2) is able to reliably assess length estimation competence based on measurement invariance.

It is remarkable that – as with other areas of mathematical abilities – Taiwanese students outperform German students in length estimation competence. However, interestingly an explorative analysis indicated that German students show a similar competence for specific items addressing not small and not touchable TBEÖs. Taking into account that in the Taiwanese mathematics curriculum and instruction, estimating measures by the use of body parts is a constant component in the Taiwanese curriculum while German students are rather taught specific knowledge about the length of some objects, this is not surprising (Ruwisch and Huang, 2018). Here, the opportunities to learn have impact on the length estimation competence.

If this result will be confirmed in further in-depth analyses then it suggests that mathematics educators need to be aware of the influences of the different estimation situations and should provide learning opportunities involving these situations separately in mathematics classes.

References


COUNTERACTING REPRESENTATIONAL VOLATILITY IN GEOMETRY CLASS THROUGH USE OF A DIGITAL AID: RESULTS FROM A QUALITATIVE COMPARATIVE STUDY ON THE USE OF PENTOMINO LEARNING ENVIRONMENTS IN PRIMARY SCHOOL

Tobias Huhmann*, Karina Höveler** and Katja Eilerts

Abstract

The discussion of digital learning in mathematics education research is especially focused on the potential digital media may hold for learning mathematics. To this end, challenges in analogue learning situations, such as the representational volatility of action processes, must be identified and then theoretically analysed and empirically tested in order to determine in what way or to what extent the use of digital aids could support analogue learning in this regard. This article presents results from a qualitative study carried out with children in the third year of primary school on the use of a digitally

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supported learning environment for pentominoes providing initial insights into how representational volatility can be counteracted by means of digital aids. The findings presented here focus on similarities and differences between processing pentominoes tasks with an analogue versus a digital learning environment for further development of the app to counteract representational volatility.

**Keywords:** geometry, primary school, learning environment, digital aid, representational volatility

**Theoretical background**

**Digital learning and teaching in primary school mathematics**

Digital transformation has been accompanied by the digitalisation of learning and teaching. This is currently a significant topic in education policy. Yet it remains unclear what really constitutes digital learning and teaching in primary school and what is the role and potential of digital media (educational software, tools, apps, etc.) in influencing educational processes (in the mathematics classroom) in primary schools (e.g. Eilerts, Huhmann, Donevska-Todorova and Henning, 2017; Krauthausen, 2012).

Beyond the generally accepted and understood human interaction with digital media, the question arises how digital media could enrich learning of particular school subjects. Research in mathematics education has identified potentialities of digital media that must be widely explored in further research to find answers to this question. The potentialities listed below have been identified so far:

- **Adjustment between action and mental operation.** The mental processes that both guide and accompany action processes require adjustment of individual (digital) action processes and perceptual and cognitive processes (e.g. Huhmann, 2013; Sarama and Clemens, 2016).

- **Cognitive relief.** Possibilities for reducing cognitive load – e.g. simulating actions, using animations, outsourcing calculable actions, documenting steps and processes – which make available further mental resources for the actual learning process (see “cognitive load theory” in Chandler and Sweller, 1991).

- **Synchronicity and interconnecting planes of representation.** With the use of (digital) media, various representational forms can be generated and interactively (through individual actions) connected with one another (almost) simultaneously (see Schmidt-Thieme and Weigand, 2015).

- **Structuring aid.** (Digital) media can support learners in structuring, restructuring, and more quickly representing objects, and it can help them while they are using the structure (e.g. Walter, 2018).

- **Multi-touch technology.** Multi-touch technology enables learners to use tablet applications with multiple fingers, making simultaneous actions and representations possible (e.g. Walter, 2018).
The challenges and limitations of analogue learning raise the question of how the challenges can be overcome and the limits expanded with the use of digital aids. In principle, this perspective – which starts with analogue learning – leads to questions about what more and what different is possible: Is something more or something different possible with digital support? If so, what would this more, what would this different comprise? We have identified the representational volatility of processes as a central challenge in analogue learning. In counteracting this, we have recognised a further potentiality for digital media.

**Representational volatility of processes: A particular challenge in learning (mathematics)**

The representational volatility of all action or thought processes, in combination with attention and cognitive load, presents a central challenge for learning and teaching: Where no documentation of a process exists, the process is no longer present or observable after it has been completed. Its representation is not stable, hence this phenomenon is called “representational volatility” (e.g., Huhmann, 2013).

The representational volatility phenomenon enjoys special attention in the fields of media psychology and media didactics (e.g., Morrison, Tversky and Betrancourt, 2000; Schnotz and Lowe, 2008). Learners have information available only for a limited time. This means that they must focus their attention on the right thing at the right time, identify relevant information from different steps in the process and keep it in working memory (or activate it from long-term memory) and connect all of these elements with one another. Representational volatility thus demands additional cognitive requirements, limits the possibility of analysis, and often leads to more time for learning (e.g., Morrison, Tversky and Betrancourt, 2000; Schnotz and Lowe, 2008).

Mayer (2008, 2014) identifies ‘Principles of Multimedia Learning’ to manage and to reduce cognitive overload, and to foster generative learning processes. They can be used as a base for the design of multimedia learning environments for further research in multimedia learning. After analysing these principles regarding representational volatility we identify especially the following principles (Mayer, 2014) to counteract representational volatility:

- **Managing Essential Cognitive Overload:** segmenting, pretraining and modality Principles – People learn better when a multimedia message is presented in learned-paced segments rather than as a continuous unit, people learn better from a multimedia message when they know the names and characteristics of the main concepts, and people learn better from a multimedia message when the words are spoken rather than written.

- **Reducing Cognitive Overload:** coherence, signaling, spatial contiguity, temporal contiguity; – People learn better when extraneous material is excluded rather than included, when cues are added that highlight the
organization of the essential material, when corresponding words and pictures are presented near rather than far from each other on the screen or page or in time.

- **Fostering Generative Processing:** navigation principles. – People learn better in hypertext environments when appropriate navigation aids are provided.

Thus, representational volatility creates particular challenges for the presentation of information, in particular for the design of learning processes: When, how long and which information is in which way represented and available for the learner during his own exploration process? – These are very important questions to ask to develop a learning environment – and it is time to ask these questions from the point of view of didactics of mathematics (Huhmann, 2013, p. 17).

Representational volatility (also) influences the learning of mathematics substantially: Every action process delivers a product of action, but the process itself is often no longer observable. In the absence of process documentation, communicating and reasoning about the individual steps in the mathematical process, about the intention of the action, about individual steps in the action, about approaches, strategies or the development of strategies, as well as the reflection of the process as a whole, become difficult if not entirely impossible, especially at the primary school level (Huhmann, 2013). This is especially relevant for the subject area of space and shapes.

Nevertheless, in the field of didactics of mathematics, this aspect has largely received no attention in the basic design of (multimedia) learning environments so far. Based on the previous comments it proves necessary to consider the aforementioned identified potentialities and the principles for multimedia design for the development of digitally supported learning environments for teaching mathematics. (How) does mathematics didactics use the findings from psychology? (ibid)

**Counteracting representational volatility in processes: Potential of digital aids for learning (mathematics)**

The central objective of the project described below is to counteract representational volatility in action and thought processes. One possible approach is to start with the limitations of analogue learning and then develop digital aids that bring about this counteraction. The authors of this article have identified a number of possibilities, including:

- Documenting processes as well as intermediate and work products, thus making individual action repeatedly accessible for analysis and reflection. The documentation is also available for further constructive and dynamic exploration (e.g. Wollring, 2007).
Problem reduction in order to counteract the complexity of action and the non-obviousness of possible actions and/or their consequences.
- Simultaneous generation of various possible products of action.
- Adaptive feedback, provided during individual actions, oriented to the action and the learner’s competence.

These aspects characterise the particular challenge of representational volatility in analogue learning in mathematics. At the same time, they are potentialities suitable to be developed into digital aids that can counteract representational volatility and investigate the mode of action in question.

**Design of the study**

**Objectives and methodology**

Within the framework of the interdisciplinary project “Digitally supported teaching and learning in the primary school mathematics classroom,” digital aids are being developed and researched as apps for selected learning environments, at present especially for the subject area space and shapes (Eilerts and Huhmann, 2018; Huhmann, Eilerts and Heinemann, 2018). The project’s overarching research question is:

*How can the use of digital media counteract representational volatility so as to address the challenges it causes in the primary school mathematics classroom?*

This provides requirements and questions at the levels of both development and research. Firstly, digital aids must be developed for the selected learning environments in accordance with the theoretical considerations, and these considerations should also be based on the principles for developing substantial learning environments (Wollring, 2007). Secondly, empirical analysis of findings on the utilisation and effectiveness of the developed digital aids should assist in both developing the apps further and developing additional digital aids aimed at systematically counteracting representational volatility in other contexts.

The project follows a design-based research approach (Cobb, Confrey, diSessa, Lehrer and Schauble, 2003; Gravemeijer and Cobb, 2006), because it systematically connects both development and research objectives. In so doing, multiple cycles of development, testing, and further development are iteratively combined. With the specifications of the challenges guided by theory, the app, “Pentominoes”, was developed in an initial design cycle and then tested in the course of an exploratory interview-based study with 26 children in the third year of primary school (age 9 to 10). The app’s potential and observable utility for counteracting representational volatility was tested according to the following guiding questions:

1. Which similarities and differences reveal themselves between processing pentominoes tasks in an analogue versus a digital learning environment and which potentialities and requirements can thus be derived for the app and
its further development with regard to counteracting representational volatility?

2. How do learners use the digital aid and to what extent is representational volatility counteracted through its use?

This article analyses and discusses the qualitative data in relation to the first question with a focus on assistance provided for “problem reduction”.

**The development of an app for pentominoes as an object of learning**

A pentomino (= square quintuplets) is a type of geometric figure, made up by conjoining five squares of the same size. A total of 12 different pentominoes can be created. They should not duplicate each other by being turned or mirrored. This object of learning comes with manifold mathematical activities: Beyond finding all possible pentominoes, the exercises notably include configuration tasks for different figures (e.g. Golomb, 1994). Such configuration exercises are designed to foster content- and process-based competencies, especially in the subdomains of visual perception, spatial thinking (e.g. in carrying out mental operations on the elements of the figure), development in visualising symmetry, mathematically communicating courses of action, reasoning, and problem-solving (see Koth and Grosser, 2010).

An analysis of principally possible executions for configuration exercises, such as a 6 x 10 rectangle, shows that these tasks are characterised by high complexity of action and by non-obvious possible actions and/or consequences of actions. Furthermore, it is not possible to simultaneously generate various possible products of action and there is neither documentation of processes nor adaptive feedback that is representational volatility is overall very high. Based on these exceptional challenges in the context of the analogue learning environment, the digital aid to be developed would have to meet the requirements detailed below.

The Pentomino-app contains different types of tasks. These include, among other things, existing and self-to-be laid/laying out tasks as well as various problem-solving tasks. For each arbitrary configuration, the software provides an algorithm that documents all possible solutions for configuring the figure, that is the entire solution space is captured by the software (for the 6x10 rectangle example, 2339 possible solutions are available). When processing a task, learners have the following options for assistance available, each of which can be activated by clicking its relevant button:

- Problem reduction by breaking down the problem into subproblems, whereby the necessary pentominoes for the solution of each subproblem are disclosed.
- Successive setting of target positions for individual pentominoes within the figure as a whole.
• Checking the positioned pentominoes with an algorithm that matches the largest quantity of correctly placed pentominoes with the stored solutions and identifying the correct pentominoes by flashing them on the display.

Data collection and data analysis
Following the theory-based development of the app, taking into account the above-mentioned requirements, exploratory pair interviews, based on particular guidelines (see Diekmann, 2008), were conducted with randomly paired learners in the third year of primary school. The 28 learners who took part were divided into two groups. Each group comprised 14 learners, making seven pairs. The approach was based on a counterbalanced design to produce no order effects related to whether the analogue or digital materials were used first. Each of the 14 pair interviews took about 45 to 60 minutes. (See Table 1.)

<table>
<thead>
<tr>
<th>Group 1</th>
<th>Analog learning environment</th>
<th>Interview</th>
<th>Digital learning environment</th>
<th>Interview</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 2</td>
<td>Digital learning environment</td>
<td>Interview</td>
<td>Analog learning environment</td>
<td>Interview</td>
</tr>
</tbody>
</table>

Table 1: Counterbalanced design of the exploratory interview study
To address the research questions, the video-recorded and transcribed interviews were analysed to prevent the backdrop of grounded theory (Glaser and Strauss, 1967) and qualitative content analysis (Mayring, 2015). Grounded theory was used to identify the core topics in an iterative process. In the process, the children who received the analogue learning condition first were compared with those who received the digital condition first.

Results
For the presentation of the results of research question 1, the analyses are carried out as an example on the laying task for the 6x10 rectangle. Similar findings could also be obtained for other laying tasks. As it turned out, none of the 14 children who started with the analogue learning condition could solve the given problem (configuring a 6x10 rectangle) without assistance.

Figure 1: Screenshot of a third of a three-part aid (6x10 rectangle)
After problem reduction, provided in this case by a three-part aid (see Figure 1), all children could solve the problem. The same results were obtained when the children used the digital media, regardless of which learning environment they received first. The main aids while processing the task with the problem reduction are: Decomposing the whole problem into subproblems, where the Pentominos required to solve each subproblem are known and visual perception (Eilerts and Huhmann, in press).

This indicates that “division of the problem into thirds” for rectangles in both the analogue and the digital versions provides constructive assistance. Principally this can be achieved in either analogue or digital versions. However, because of how the app is programmed, the digital version holds the possibility to provide such a problem reduction for all possible solution variants (e.g. 2339 possible solutions in the 6x10 rectangle) at “the press of a button.” In contrast, offering such assistance in the analogue version might be possible in theory, but very time-consuming in practice. It would thus only be realisable for example problems and consequently not adapted to each learner’s individual approach to solving the problem. After 15 minutes of working on the task, the children seemed especially frustrated with the analogue version; no such frustration was apparent for the digital task. It can be assumed that the oft-cited novelty effect when introducing digital media is at work here (Herzig, 2014). Dividing the problem into thirds did not suit individual approaches in all learning environments and therefore also did not address the support the learners needed at an individual level. Some learners had, for example, already placed some pentominoes correctly, but the division of the problem into thirds could not respond to these already solved elements; instead, it offered a reduction of the problem as a whole that was unrelated to the individual’s approach. In response to the question of what kind of help they would like, Fatima, for example, said, “So that I can find out what I need to change.” Some learners said they would like to know which of the pentominoes they had already placed were not yet correct; others wanted to know which were correct.

The individual assistance that the children desired is not practicable in the analogue version, because that would mean that all possible solutions for the chosen configuration must be available and that the teacher would have to check which of the learner’s correctly laid pentominoes in the figure agree with which provided solution. However, for the programming of the app, this is possible. It is thus possible to derive new requirements for the further development of the app. Adjustments that take into account the individualisation of assistance need to be carried out, making it possible to

- segment the figure in any way desired in accordance with each learner’s individual solution approach and
- provide feedback on the correctly placed pentominoes that is oriented to each individual approach.
The next step in development is thus to use a configurable support system to develop an individually and dynamically changeable problem reduction. This would take into account two aspects. Firstly, the problem as a whole would be dividable into subproblems as desired, whereby the pentominoes required solving a subproblem would be (i) disclosed or (ii) not disclosed. Secondly, an adaptive feedback mechanism oriented toward the individual learner’s approach to the problem would reveal correctly placed pentominoes when a button that activates checking is clicked.

Conclusions

The first results of this qualitative study show that one aspect of representational volatility can be successfully counteracted by the developed digital aids, namely problem reduction. In regard to adaptive guidance provided by the developed app, development needs could be determined and put in concrete terms for future work. We see our fundamental position, which emanated from the analysis of challenges and limitations of analogue learning (in mathematics) and the potentialities for digital aids it holds (in short: conducting potential analyses), and the development and research conducted on this basis as validated. Considering the continuously expanding market for apps, a constructive yet critical perspective is called for in this regard at all times.

References


MISCONCEPTIONS ABOUT THE RELATIONSHIP BETWEEN PERIMETER AND AREA

Darina Jirotková, Paola Vighi and Renáta Zemanová

Abstract

The paper describes partial results of a research conducted in primary schools with 10-11 year old children. The research focused on the issue of the relationship between perimeter and area. The main goal is to study the presence of a misconception documented by a number of researches and named “same A, same B”, that is, in our case, “same area, same perimeter”. Thinking about the relationship between area and perimeter of 2D geometrical shapes and the need to articulate the thoughts gives pupils the opportunity to grasp these two concepts deeper. The results show how pupils understand the geometrical concepts of area and perimeter, which are part of the topic ‘Geometry and measurement’, which is one of the key fields of school mathematics. Suggestions on possible improvements of didactical approach to this topic are presented.

Keywords: geometrical thought, visualization, juxtaposition of shapes, perimeter, area

Introduction

The presented research stems from one question of assessment posed by INVALSI (Italian National Institute for the Evaluation of Instruction and Formation Educational System). The authority organizes periodic and systematic tests on pupils’ knowledge and abilities in Italian language, Mathematics and English language. The tests are submitted in all Italian schools to pupils in the 5th grade of primary school (10-11 years old) in the same day, at the end of the school year.

In our research study, we focus on task ‘D11’ from a battery of 35 tasks/questions with different mathematical content.1 Pupils were asked to solve these 35 questions within 75 minutes. A part of these questions is multiple-choice questions. Another part are open questions to be answered in writing: pupils are expected to explain their reasoning. Sometimes the task consists of drawing. Pupils are allowed to use any tool for drawing, but not a calculator.

Task ‘D11’ was suitable to become our research tool as its solution can reveal the well-known misconception “same A, same B”. To inquire deeper it, we decided to design a research study2 at Charles University in Prague, involving Czech and Italian researchers, which would be implemented in primary schools in both countries.

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1 More detailed information see: https://invalsi-areaapprove.cineca.it
2 EU project Investigation of the development (of mental schema) of geometrical pre-concepts and concepts in pupils of age 5 – 10 years, No CZ.02.2.69/0.0/0.0/16_027/0008495
The problem and its a-priori analysis

Task ‘D11’ first presents a particular trapezium and then pupils are asked to compare areas and perimeters of three geometrical figures created by two trapezia congruent to the first, connected in three different ways.

Task ‘D11’

Look at the right-angled trapezium below.

\[ AD \text{ measures the double of } BC. \ BC \text{ is equal to } AB. \]

All the figures below were obtained using two trapezia equal to \( \triangle ABCD \).

Complete both spaces by one of the two words written below the dotted lines.

The three figures have \( \text{area and perimeter} \)

\[ \begin{array}{c}
\text{(equal/different)} \\
\text{(equal/different)} \\
\end{array} \]

Figure 1: Task ‘D11’

Task ‘D11’ presents a long text, which some pupils might find difficult to read. The first part about the relationships among the sides of the trapezium is complicated and not usual in schools. Usually problems present measures of sides. Moreover, the first part is not needed to solve the problem, since the comparison of the areas could be trivial and the comparison of perimeters could be made just by looking at the pictures (and may be using hands) and some argumentation.

The task is unusual from the perspective of traditional approach for several different reasons:

- the first trapezium is drawn in an uncommon position, traditionally the ‘bases’ are ‘horizontal’;
- three figures are presented, usual geometric tasks consider only one figure;
- each of the three figures is created by combining two trapezia (putting one side to one side without overlapping);
- figures (a) and (b) are convex polygons, figure (c) is nonconvex. Traditionally both Italian and Czech primary school teachers work with convex polygons only;

- the final question asks pupils to compare simultaneously area and perimeter. Traditionally teachers in Italy work with perimeter in the 3rd–4th grade and with area in the 4th–5th grade. In Czech schools both concepts are usually introduced in 4th grade, mostly together with formulas and procedures how to determine both.

The reason why in task ‘D11’ the comparison of areas comes before comparison of perimeters could be motivated by the following ideas:

- the comparison of areas can be easier than that of perimeters
- the need to observe if pupils use the misconception “same areas implies same perimeters” (see ‘Theoretical framework’).

The way the three figures are drawn can initiate some incorrect reasoning such as:

- figure (a) has a greater area since it is higher than the other figures
- figure (b) has a greater area since it is wider than figure (a) and perceptively more compact than figure (c).

Task ‘D11’ was submitted to the pupils in Italy by INVALSI in May 2017. The results were the following: 46.9% correct answers, 50.9% incorrect answers and 2.2% other. It is necessary to take into account that task ‘D11’ is one of 35 question of the test, which means Italian pupils did not have a lot of time to answer this question. Moreover, the question is a “double question” about area and perimeter, so pupils could have given a correct answer about area and incorrect about perimeters or vice versa. In that case, the answer is considered incorrect globally.

The task seemed very interesting to us for two reasons: a) the simultaneous involvement of two concepts – perimeter and area, b) the topic ‘Geometry and measurement in 2D’ is considered as a critical part of mathematics both on primary and secondary school level (Rendl and Vondrová, 2013) by teachers. Moreover, in Czech there is a linguistic problem that is an obstacle to good understanding.3

For these reasons, we decided to use the task ‘D11’ in our research and to study, in particular, the above mentioned misconception.

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3The Czech term ‘obsah’ for the area in geometry refers to content in everyday life language – content of a pocket, of a bottle, of a book. The word ‘obsah’ is not used in everyday language for the geometrical concept of area. Instead, the words ‘plocha’ (surface) or ‘výměra’ (expanse) are used. Also the word ‘obvod’ (perimeter) is used in everyday language but has a different meaning than in geometry – circuit or waist length.
Theoretical framework

In school geometry, the concepts of perimeter and area are usually introduced hand in hand with their measurement and formulas, without previous development of the concepts themselves.

“In our opinion the early introduction of measurement in geometry presents the risk of confusing a g-quantity with its measure (q-quantity), whereas it is in fact necessary to emphasize that an object has a length or a surface, even if these are not measured.”4 (Marchetti et al., 2005)

In other words, in teaching these topics it is essential to allow pupils to discover that length or area are properties of objects that can be considered without numerical descriptions. For instance, it is possible to compare areas of two surfaces by manipulating with their copies cut out of paper, putting them one on the other, and to compare lengths of two curves using cords.

The previous argumentation builds on the distinction made by Skemp (1976) between “instrumental” and “relational” mathematics:

“… instrumental mathematics is easier to understand and it provides quick results (sometimes ‘apparent’). The relational mathematics is more adaptable and it allows to develop argumentative and logic competencies.” (Vighi and Marchini, 2011, p. 393).

The consequence of an untimely introduction of measurements can initiate a shift from the geometrical to the arithmetic field. To verify equivalence of surfaces or perimeters, pupils compare the numbers that express their measure (Chamorro, 2001). A problem connected with too early activities with measurement is that it can hinder construction of the notions of perimeter and area. For instance, the use of grids to estimate or evaluate areas can block the real nature of this concept (Jirotková and Vighi, 2016). Obviously, measurement is important, but it is also necessary to propose activities that involve both these concepts without measuring:

“While most researchers concentrate on the enumeration of units as the main notion in the learning of area and volume (Outhred and Mitchelmore, 2000; Battista, 2004; Sarama and Clements, 2009), Battista proposes a learning trajectory for area and volume that contains two parallel streams: measurement and non-measurement reasoning.” (Tůmová and Vondrová, 2017, p. 101).

These are the reasons why we decided to present to pupils in our research a worksheet based on ‘D11’, asking them to give the answer without using ruler.

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4 We used g-quantity to signify the Italian word grandezza that emphasizes the qualitative aspects of some mathematical entities.
It is also common in our schools that the concepts of perimeter and area are applied only to basic shapes: squares, rectangles, triangles. This may result in the following:

“A common misconception that impedes learning includes the beliefs that shapes with the same perimeter must have the same area.” (Geary et al., 2008, p. 4-xxii)

and, in our opinion, also vice versa.

Many researchers studied the topics ‘perimeter’ and ‘area’ and pupil’s mathematical reasoning or misconceptions about them. In particular, starting from the premise that pupil’s ideas are often not in line with scientific thinking, Stavy and Tirosh (1996) write that pupils tend to use intuitive rules. In particular, they show that pupil’s conceptions may come from the same alternative rule, which these authors call “more of A, more of B”. Murphy (2010, p. 1822) evokes this rule in her study on the misconception related to perimeter and area:

“The recognition of such a misconception goes back at least to the 1960s with Lunzer’s (1968) notion of ‘false conservation’. This false notion has more recently been cited by Stavy and Tirosh (1996) as an example of the intuitive rule ‘more A, more B’, in that as the perimeter increases so the area will increase. Alternatively the intuitive rule can be manifested as ‘Same A, Same B’ in that the same perimeter will mean the same area.”

In our experiment, we suppose that the authors of task ‘D11’ wanted to see if the misconception “same A, same B” appears in the form “same area implies same perimeter”. It is a very strong misconception, which emerged also in some researches involving teachers (Vighi and Marchini, 2011).

The main research questions of this study are:

- Facing the task ‘D11’, are pupils tricked by the idea “same areas, same perimeters” or are they not?
- Which misconceptions do appear in answers to the question?

In other words, starting with task ‘D11’, we want to conduct a qualitative research study on intuitive reasoning used by pupils, with the aim of investigating their misconceptions and their impact on mathematics education.

Methodology

The research was conducted in two countries, the Czech Republic and Italy, with 10-11 year old pupils. The Czech research sample consisted of 10 classes, 182 pupils, the Italian sample of 4 classes, 97 pupils, chosen in ordinary state schools in Prague, Ostrava (CZ) and Parma (IT). The whole classes participated.

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5 Many thanks to Italian teachers M. Morini, B. Riccardi, L. Spinazzi, and to Czech teachers S. Holáková, M. Hálová, E. Ždímalová, E. Janšová, A. Čmoková, H. Raszková, E. Paličková, R. Ignáciková, M. Delongová.
First, we presented the worksheets to the pupils, explaining they must not use any ruler to measure the sides of the shapes involved. The working time was approximately 15 minutes and could be extended. Then we collected the worksheets and analysed pupils’ solutions using techniques of grounded theory (Glaser et al., 1967). Some days later, we asked teachers to initiate a class discussion on the worksheet, analysing the possible answers. Subsequently we organized another activity based on pupils’ own creation of shapes using two trapezia equal to \(ABCD\) and on their classification (see Vighi and Jirotková, 2019).

**Results**

The majority of pupils recognized that the areas of the three figures (a), (b) and (c) drawn in the worksheet (Figure 1) are equal, without difficulties. Their explanation was this: each of the three figures is composed by connecting two trapezia equal to trapezium \(ABCD\), so the shapes change but not the areas. Sometimes they also illustrated how to transform one shape into another to obtain three equal shapes. This confirms that at this age the conservation of quantity is stabilized in most cases. Nevertheless, sometimes we found answers such as “Figure (b) is tubby, so its area is bigger” or “the areas are different since one shape is longer, another larger”. Sometimes they used natural language and metaphors to describe the shapes: Figure (a) is a vase or a vertical boat, Figure (b) looks like a house or an arrow that points to the top, Figure (c) a slanting arrow or a sock or letter L.

As to the perimeters, the situation was different and more articulated. The percentages recorded are very similar in both the countries: the wrong statement “same areas and same perimeters” (upon middle cell) reveals respectively 42% of answer in Italy and 40% in Czech Republic; the statement “shapes have the same areas and different perimeters” (bottom middle cell) is correct for the 54% of Italian and 53% of Czech pupils. In the following table we report the numbers of the results (we disregard omitted or unclear answers).

<table>
<thead>
<tr>
<th></th>
<th>Same areas</th>
<th>Different areas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Same perimeters</td>
<td>N = 32 (IT)</td>
<td>N = 1 (IT)</td>
</tr>
<tr>
<td></td>
<td>N = 62 (CZ)</td>
<td>N = 7 (CZ)</td>
</tr>
<tr>
<td>Different perimeters</td>
<td>N = 41 (IT)</td>
<td>N = 2 (IT)</td>
</tr>
<tr>
<td></td>
<td>N = 83 (CZ)</td>
<td>N = 4 (CZ)</td>
</tr>
</tbody>
</table>

Table 1: Summary of answers to task ‘D11’

We must clarify that the numbers in Table 1 do not provide a proper overview of the real situation, since they show numerical data without taking in account the explanations given by the pupils. For instance, the correct answers (same areas, but different perimeters) are very often associated with wrong argumentations. We classified them in this way:
a) not relevant (e.g.: Each figure is concave or convex, high or low, so all the perimeters are different);

b) denoting a misconception (e.g.: To compose the figures two equal trapezia are used, so the areas and perimeters must be equal);

c) showing weak communication skills (e.g.: The trapezium is always the same).

Communication skills are developed as the need to articulate ideas. Weak language usually suggests weak thoughts.

As to the areas, to prove their equality some pupils showed, using drawings, how to transform each figure into another. Therefore, they employed the ‘equal decomposition’ of the figures. In the written answers, we noticed also the use of different words such as ‘equal’, ‘congruent’, ‘equivalent’, sometimes in an incorrect way.

As to the perimeters, we found the appearance of the misconception ‘same pieces, same perimeters’ in 21% of the answers, with the following explanation: Changing the position of the two trapezia, also perimeters change (so, the misconception ‘same A, same B’ appears as ‘different A, different B’). In addition, the misconception connected with a wrong estimation of the length of the ‘oblique side’ of the trapezium is present: The perimeter of figure (a) is 8, of (b) is 6 and of (c) is 8.

**Examples of answers related to areas**

Areas are the same because of the use of the same trapezia:

- The shapes are all the same but put differently.
- I don’t need to know the length of the sides because each of the figures is made of two trapezia. The trapezium is always the same. So, the area will be the same.

One pupil also pays attention to the symmetry of the two trapezia in each figure:

- Because they are two identical figures. In each figure, one of the trapezia is symmetrically rotated.

The ideas are good but their formulations are weak:

- I think the area is the same because they are the same anyway.
- Each has one half of the house.

One pupil felt the need to find a more profound argument, so he expresses the area by the number of congruent squares.

- Figure ABCD is in fact a square with a triangle attached, which is one-half of the square. If I take the trapezium twice, I get twice a square and twice a triangle. If I join two halves of a square, I get one whole square. So I have 3 squares.
Sometimes the pupils tried to explain the equality of areas using drawings, showing that the three figures can be transformed into the same figure, for instance a rectangle.

**Examples of answers related to perimeters**

Some pupils were able to explain why the perimeters were different:

- *Perimeters are different because the trapezia are not connected in the same way.*

- *In figure (a) side AB disappeared, in figure (b) side AD, in figure (c) side DC, so in each shape different sides are removed. This is the reason why they have different perimeters.*

We believe the first statement is an example of a good pupil’s idea that is badly expressed by words. The second statement highlights the main feature of problem ‘D11’: the perimeters could be compared by observing the trapezium sides that “disappear”, instead of the boundaries. This strategy can be simpler, but it needs an unusual reasoning.

Another explanation is: *The three figures don’t have equal perimeters, since they have different sides.* In this case, the answer is correct, but we cannot be sure about the correctness of the reasoning: maybe the pupil did not use the information on the sides of trapezium $ABCD$, but only visually investigated the boundaries.

The majority of pupils for whom “perimeters are equal” explained it in this way: *The three figures have equal perimeters, since they are composed of equal trapezia*. So, the misconceptions “same A, same B” appears where A is ‘trapezia’ or ‘shapes’ and B is ‘perimeters’. In other words, the perimeters are considered equal as the pieces that form the three figures are the same, although in different positions. This implies that the conservation of quantity is applied also to the perimeters. The comparison of perimeters is based on the same reasoning made on areas.

A misconception of the kind “same A, same B” appears also in the statement: *The perimeters are different since the figures are different*, so different shapes seem to imply different perimeters.

Sometimes the difference of perimeters is derived from the differences of the figures: *the first and the second have slanting sides, the third only straight sides*.

A pupil tried to use an approximate measurement: *Perimeters are different since AD is two times AB, AB is equal to BC and CD is one time and a half of AB, so the perimeters measure respectively 9, 7 and 8*. In other words, the pupil assigned length 1 to $AD$, 2 to $AB$ and 1.5 to $CD$. In fact, we know that the ratio between lengths $CD$ and $AB$ is an irrational number, but at this level the answer could be accepted.
Some protocols document another misconception: the length of $CD$ is understood as the same as the length of $AB$. It is a misconception documented in literature (Vighi, 2006), maybe caused by the use of the grid paper: to find out the length of a ‘slanting’ segment, pupils identify it with the length of its major vertical or horizontal projection. In our case, projections of $CD$ are equal, and they are as long as $AB$.

Some answers are supported by different interesting arguments. Although the use of a ruler was forbidden, a pupil tried to use measurements: *In figure (a) there are four short sides, I thought that their measure could be 5 cm, so 20 cm; AD is the double of BC, so its measure is 10 cm. 10x2=20, so 20+20=40 cm. I made the same for the other figures.* In fact, the pupil calculated only the perimeter of the first and he transferred the result to the other two figures. Another child tried to overcome the missing ruler by counting the sides, even if they have different lengths: *Perimeter is always 6.*

Very often correct answers were accompanied by incorrect or awkward arguments such as: *The sides are equal, but posed in different ways, so perimeters are equal since as in addition, when I change the order of the addends, the result doesn’t change.* The pupil used reasoning from addition (commutative law).

**Conclusion**

The results of the research study open many issues. As to comparison of the areas, the number of ‘correct answers with incorrect explanations’ could be justified by the difficulty to explain the reasoning by words, because the visual information is very strong. We can say that these pupils could be, from the cognitive point of view, half way between the 2nd to the 3rd level of Van Hiele’s stages of understanding geometrical phenomena (Van Hiele, 1986). Their communicative skills could be insufficient to articulate their ideas in a comprehensible way.

As to comparison of the perimeters, the misconception “same pieces, same perimeters” had strong influence on the answers. The reasoning used for areas successfully was applied again for perimeter. It allows the solver to faces the problem without any need of deep reasoning or, following Stavy and Tirosh (1996), it could be the case that sometimes the relations between A (area) and B (perimeter) were perceived as qualitative only. The reason could be also that the visual perception of the area is dominant and overrides visualization of the perimeter. Another possible explanation is that the word ‘perimeter’ is used with different meanings: it could indicate the boundary, the length of the boundary and also the measure of the length of the boundary.

The study confirms that forbidding the use of the ruler put the pupils to a really difficult situation. However, it was necessary for the research aim. It confirms also the need to build the concept of perimeter by two kinds of activities: first, by comparison of lengths of boundaries of shapes without measuring, and only then
with the use of numbers for measuring (no matter which unit of measure is used). In this way, the initial geometrical problem is transferred into the arithmetical field only after the construction of the geometrical meaning of perimeter.

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GETTING TO KNOW SHAPES OF THE SAME AREA – HOW PUPILS BUILD ON ONE ANOTHER’S WORK

Jaroslava Kloboučková and Nad’a Vondrova

Abstract

The study is part of action research aiming at the description of concept development in mathematics. Building on socio-constructivist theories of learning, we pursue the question “How do pupils get to know that shapes which do not appear congruent can have the same area, by building on one another’s knowledge in a classroom situation?”.

A qualitative design is used in the study. Its participants are Grade 1 pupils of a Prague primary school. Through the identification of several episodes, we show how quite an unexpected discovery of one pupil in Grade 1 influenced the knowledge construction process of the above knowledge for the whole class. We describe how the class developed shared knowledge and how the teacher influenced the whole process.

Keywords: substantial learning environment, concept development in mathematics, congruent shapes, paper folding, theory of generic models, area

Introduction

In order to design teaching and learning processes, it is necessary to find out how concepts are built in pupils’ mind. The first author, using action research paradigm, systematically followed the work of her pupils from Grade 1 to Grade 9, as they were struggling with mathematical concepts. She used a schema-oriented teaching strategy based on developing mental schemas (here VOBS) (Hejný, Jirotková and Slezáková, 2007; Hejný, 2012), a specific constructivist way of teaching.

[The method] is based on respecting 12 key principles, which it combines into a coherent approach enabling children to discover mathematics by themselves and with enjoyment. It builds on 40 years of experimental work and puts into practice
One of the tenets of the method is that pupils build on one another’s ideas and thus, class discussion is at the heart of teaching. The article presents how pupils gradually built the idea that shapes which do not seem to be the same can have the same area already in their first grade.

It is commonly acknowledged by research that pupils have problems with grasping the concept of area generally and the concept of area conservation in particular (e.g., Kamii and Kysh, 2006), often hold a misconception that only congruent shapes can have the same area or “the same area the same perimeter” misconception (e.g., Kospentaris, Spyrou and Lappas, 2011). One way of preventing this misconception is by using such activities in the classroom which lead to the Euclidean method of area comparison also known as additivity axiom and the principle of overlapping as a general method to compare areas of two shapes (Zacharos, 2006). Manipulation with cut-out shapes as well as paper folding are natural means through which the two principles above are observed.

**Theoretical framework**

Our work is positioned within a concept development theory of generic models (Hejný, 2012). Novotná, Tichá and Vondrová (2019, pp. 199–200) describe it:

[The] theory describes concept development […] as consisting of several levels, beginning with motivation, through the stage of isolated models (concrete cases of future knowledge) and the stage of generic models (which comprise all isolated models and can substitute for them) up to the abstract knowledge level. There are two shifts between the stages: generalisation and abstraction. The latter is accompanied by a change in language […]. Crystallisation [describes] […] connecting new concepts to old ones and using it to build new knowledge […]. Within this theory, insufficient understanding is captured by the term mechanical understanding, which means knowledge that is not supported by generic models and is mostly grasped by memory only.

VOBS ascribes to the socio-constructivist theory of learning which stresses that new knowledge is developed in cooperation among pupils. Learning “is not regarded as restricted to individual processes proceeding in the single individual, but it is perceived essentially as a process of active construction of knowledge by the learners in interactive instructional settings” (Steinbring, 2005, p. 62). Hejný (2014) introduces the term *cognitive osmosis* for the process in which one pupil/group makes a discovery and shares it with the class which leads to the apprehension of the new knowledge by other pupils.

In VOBS, mathematical knowledge is developed by pupils solving carefully developed tasks often set within didactic environments. These environments comply with the requirements of substantial learning environments (Wittmann,
2001, p. 2). We will illustrate it by an example of the environment of Origami which is in the centre of our attention here.

(1) “It represents central objectives, contents and principles of teaching mathematics at a certain level.” Origami is used for the introduction of geometric shapes, their properties and relationships among them already in Grade 1.

(2) “It is related to significant mathematical contents, processes and procedures beyond this level, and is a rich source of mathematical activities.” Tasks within Origami are manifold and set the ground for concepts such as symmetry, area (paper folding reinforces seeing geometric shapes as parts of a plane of a certain surface not only their boundaries), fractions and even algebra.

(3) “It is flexible and can be adapted to the special conditions of a classroom.” Various tasks can be set within Origami, of a different difficulty (graded tasks).

(4) “It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research.” Origami with its graded tasks can be used both for individual and group work. Manipulation provides support to pupils when solving cognitively demanding tasks which they would not probably be able to solve without it – they first create knowledge in action. Manipulation enables pupils to show how they think about the shapes and to persuade others about the validity of their claims. These events are particularly useful for getting an insight into pupils’ reasoning.

Our research question is: How do pupils get to know that shapes which do not appear congruent can have the same area, by building on one another’s knowledge in a classroom situation?

**Methodology**

The study is a part of longitudinal action research (Breen, 2003) aiming at concept development in mathematics which has been carried out by the first author in her class in the last 9 years. The methods particular to action research were used to gather data to describe how pupils learn concepts through their work in didactic environments. Many lessons were video-recorded, artefacts from the lessons (such as pictures of the class work on the black board) and pupils’ written artefacts (such as copies of their exercise books and other written work) were collected. The teacher wrote reflective diaries of her lessons and conducted formal and informal interviews with pupils to get more insight into the way they reasoned.

The participants of the study presented here were Grade 1 pupils from a Prague primary school mostly attended by pupils from the neighbourhood. There were 19 pupils in the class, four of them had special needs.

In view with our research question, we searched the first author’s diaries for references to tasks related to the idea that shapes which do not appear at first sight to have the same area indeed can have the same area, repeatedly watched relevant
video-recordings of lessons and marked the appropriate episodes in them, and studied pupils’ artefacts to trace down stages through which the class went on the way towards the above knowledge. When possible, we traced the development of a particular pupil, mostly the pupil who contributed to the discovery of new knowledge. At the same time, we made a priori and a posteriori analysis of tasks set within the environments used in VOBS which potentially lead to the knowledge in question.

**Results**

**Episode 1: Jan finds two trapezia with the same area (September, Grade 1)**

The goal of the lesson in which the episode was observed was to provide pupils with the first experience with dividing a square into halves. They were given two paper squares and their task was to fold them into two equal parts. Some pupils folded the square along the diagonal (thus creating congruent isosceles triangles), others folded it along the axis of the opposite sides of the square (thus creating congruent rectangles). In this way, pupils as a class made a lot of isolated models (Hejný, 2014) of the division of a square into congruent parts.

During the work, the teacher noticed that Jan had problems with fine motor skills and was not able to fold the square along the diagonal precisely as he wanted. His folding led to two right-angled trapezia, the shape not known to the class yet.

Jan: It does not work. *(He nearly cries.)*

Štěpán: Let me see *(looks at Jan’s square)*. But it is correct. *(He folds and unfolds the square.)* […]

Marie: *(She shows how she made two congruent triangles.)* One must do it precisely, so that they are triangles. Jan did not succeed in it!

Štěpán: But I think that Jan is also correct even though he does not have two triangles.

The teacher asks him and Jan to come to the board and show it to the class.

Štěpán: I think that it is the same but they are not triangles, but some kind of little squares, not squares, but … *(he is searching for a suitable word; the class does not seem to agree).*

Vašek: But if it sticks out, it cannot be the same, some part is missing here.

Štěpán: But the same part is extra here.

The teacher asks the class who agrees. Most of the pupils do not agree. Jaroslav puts his hand up indicating that he does and the teacher asks for his explanation.

Jaroslav: I would cut it out like this. *(He cuts the square along the fold.)*

Štěpán: *(He puts the two parts one on top of the other and faces Jan.)* See? It is the same.
The teacher praised them and mentioned that they made “a beautiful right-angled trapezium” but otherwise did not elaborate on the above ideas. According to her, Štěpán’s argument supported by concrete manipulation was informative enough for the others to grasp it.

The extract illustrates how pupils gradually built new knowledge by using each other’s ideas. It seems that Štěpán was able to discover a generic model of how to divide a square into congruent shapes (trapezia) and at least Jaroslav discovered a generic model of how to check whether two shapes are congruent by overlapping. The question is whether sharing their reasoning and presenting other pupils with their generic models influenced in some way the rest of the class.

Next, the pupils were asked to use the second paper square and fold it again into two congruent parts, different from their first one. The teacher’s original goal for this second part of the task was to enrich pupils’ repertoire of isolated models. But given the episode above, she was also curious to see to what extent Jaroslav’s and Štěpán’s ideas were grasped by others. The pupils were indeed inspired by their classmates’ work and were trying to make two trapezia. Not all of them succeeded, though, as it was necessary to observe a rule which was not mentioned by the two boys: the distance of the fold from the side of the square must be the same on both sides. Most of the pupils persisted and experimented until they were satisfied with their result. Different folds were found (see Figure 1).

![Figure 1: Pupils’ attempts at folding the square into two trapezia](image)

While individual pupils discovered at least two ways of folding the square into congruent parts, we can hypothesise that the class as a whole, by discussing and observing one another’s work, reached the generic model of the concept. They could see that there were many (even infinitely many) ways. The teacher made a mental note to revisit the task later.

**Episode 2: Making a square (January, Grade 1)**

Pupils were given a worksheet (Figure 2) with a square and ten parts of the square, whose area was half of the area of the square. Pupils were given the same shapes cut out from paper. Their task was to colour the shapes which make a square in the same colour.
The video-recording shows that while most of the pupils used the cut-out shapes to experiment and find out which two fitted together, at least four pupils first decided which two made a square and only afterwards checked it by manipulation. All pupils coloured two rectangles and two triangles correctly. About half of them coloured all pairs of shapes correctly. Further five pupils selected one pair of trapezia but failed to colour the other one.

Figure 2: Parts of a square

Episode 3: Folding the square revisited (May, Grade 2)

In the next grade, the pupils were asked to find several ways of dividing the square into two congruent parts again. This time, they were to cut out the shapes and then compare the parts they made with those made by their neighbour at the same desk. By asking them to cut, the teacher wanted to evoke the idea of checking the congruence by overlapping without actually saying it to the class.

The pupils worked enthusiastically and soon started comparing their shapes. If the two pupils cut out the same shapes (e.g., triangles), it was easy for them to decide that the shapes were the same. In other cases, they argued about the size of the two shapes using different arguments. For example, if their shapes were a rectangle and a triangle, they argued that the triangle was bigger as “it has a tip here and it sticks out over the rectangle”. But if the pair had two different trapezia or a trapezium and a triangle, they could not decide which is bigger.

After some time, the pupils found some trapezia which had the same size but were lost as for the others. The teacher divided them into groups and each was given a rectangle, a triangle and a trapezium. One of the groups in which Štěpán worked claimed that the part of the triangle “which sticks out over the rectangle” can be cut out and placed on the rectangle (see Figure 3). In this way, they, in fact, showed by overlapping that the two shapes had the same area (the teacher did not mention area yet). When the group explained their reasoning, the rest of the class did the same for their shapes. This naturally led to new experiments – pupils tried to find out which part of the trapezium should be cut out and re-positioned to show that the trapezium had the same area as the other two shapes.
At the end of the lesson, the pupils evaluated their work and expressed joy that they found the shapes of “the same size” even though they “looked different”. The teacher asked them how it was possible. Some pupils realised only then that if all of them had the same squares before cutting them into two halves and if they fulfilled the task correctly, they made two parts of the same “size” (area) and thus, it must be the same for all shapes which originated from the square in this way.

To sum up, this activity provided pupils with more opportunities to explore the idea of shapes of the same area. The knowledge built earlier was further consolidated and expanded by a focus on division of a rectangle and a triangle into parts so that a new shape of the same area originated.

**Intermezzo**

Other Origami tasks were used such folding paper into fourths, creating ornaments and various envelopes, or dividing a rectangle into congruent parts. For example, in Grade 2, the pupils learned about fractions via paper folding (Figure 4).

![Figure 4: Using paper folding for fractions](image)

Translation: One third of the rectangle is blue, one third is red. How long is the rectangle? Divide it into thirds and colour each in a different colour.

The pupils also solved tasks in which they were to make new shapes from a given shape. For example, quadrilaterals were made from cut-out triangles with different sides by placing them next to each other. Other shapes than squares were folded or cut into pieces and new shapes of the same area were made (see Figure 5 for an example). By manipulation, the pupils further consolidated the knowledge that differently shaped shapes can have the same area.
Episode 4: Dividing a square into quarters (Grade 5)

In Grade 5, pupils continued their work with basic fractions and were asked to suggest representations of fractions. In one of the tasks, they were given a worksheet with six congruent squares and their task was to represent graphically the division of these squares into halves, thirds, fourths, etc., so that each division was different. Figure 6 shows two pupils’ solutions for quarters.

All pupils divided a square into four congruent squares and congruent triangles. Almost all were able to find two more solutions. What is important for this study is that more than half of the class divided a square into two trapezia. Thus, at least for them, the knowledge built in their first grade was firmly present in their cognitive structure. They were even able to use it in another context – fractions.

Discussion and conclusion

We showed on an example of a piece of knowledge how a class of pupils builds on one another’s work. The idea that shapes which appear different at first sight may have the same area was gradually developed by cooperation and became
a piece of “shared knowledge” (Hershkowitz et al., 2007) by the class or at least by the part of the class which was able to apply it later in another context.

It appears that the piece of knowledge, discovered (quite unexpectedly) by one or two pupils in Grade 1, influenced the others in that they were all inspired to try it out themselves. Even though not all the pupils made the discovery, they were able to take it over from the two pupils and experience joy from it. At least some of them interiorised it to such an extent that they were able to apply it in another context. This principle of shared knowledge accounts for the frequent objection towards teaching such as VOBS that not all pupils are able to solve the tasks and discover new knowledge.

In the series of tasks above, the original isolated models of the knowledge “shapes not congruent at the first sight can indeed be congruent” made by the division of a square, were augmented by other models originating from folding and cutting out other shapes and manipulation with the parts to make new wholes. In this process, in Hejný’s (2014) term, the piece of knowledge discovered earlier was connected to other pieces of knowledge, the knowledge crystallised and the cognitive scheme was enriched. Hershkowitz et al. (2007) speak about consolidation which is “a process by which the construct becomes progressively more readily available to the learner” (p. 45). This can be seen in the above episode in which the knowledge is applied in a different context. Of course, action research does not enable us to see whether this was true for each single pupil (or indeed, for each group of pupils working together on the tasks), based on the pupils’ behaviour and responses we can only make such a conclusion for a class. Moreover, the process is not completed. The knowledge becomes basis of new constructions, e.g., when formulae for the area of shapes are developed.

The teacher’s role was subdued in the construction process. She started it by setting tasks and supported it by asking pupils to present their reasoning to others. She was also able to spot an important moment of an unexpected discovery and use it to further develop pupils’ knowledge. This ability to notice in-the-moment (Mason and Davis, 2013) and act accordingly is an important prerequisite of a responsive teaching required within VOBS. The teacher acts as a facilitator and looks for ways of giving pupils space to make and share discoveries. Hershkowitz et al. (2007) investigated which interaction in the classroom lead to a joint construction of knowledge and in what ways. In our study, these were mostly interactions originating by the teacher’s invitation to pupils to present their work to others and to explain and justify it to the rest of the class.

Our study has naturally its limitations which basically lie at the heart of action research. The classroom situation is very complex, and the teacher has to observe many different aspects, the research aspect is only one of them. Thus, the description of the knowledge construction process is not as complex and perhaps deep as it would have been in case of a clinical experiment. On the other hand,
action research enabled us to observe mutual interactions among pupils and get an insight into the teacher’s intentions.

**Acknowledgment:** The paper was supported by project PROGRES Q17 *Teacher preparation and teaching profession in the context of science and research.*

**References**


MATHEMATICAL GAMES CAN MAKE A DIFFERENCE: AN INTERVENTION FOR CHILDREN AT RISK

Kerstin Larsson ☞, Paul Andrews and Inger Ridderlind ☞

Abstract

Children in public care are at risk not achieving academically to their potential. To support this group, Letterbox Club is an intervention program sending parcels with e.g. mathematical games to the children hoping this would increase their engagement to, and skills in, mathematics. We report of the effects of such an intervention where the LBC members were compared to their peers by pre- and post-test design. The over-all test results demonstrated promising effects, but no significant differences. However, certain tasks, especially subtraction tasks, stood out as LBC members had significant lower scores in the pre-test. There were also tasks where the LBC members improved significantly better than their peers. These promising results call for more studies of the effects of mathematical development by sending mathematical games to children in care.

Keywords: children at risk, intervention, mathematical games, arithmetic

Introduction

Children in public care are at risk leaving school with poor grades and less optimistic future prospects than their peers (Berlin, Vinnerljung and Hjern, 2011). Leaving school with poor or incomplete grades is a predictor for psychosocial problems such as criminality, suicide behaviour and drug problems. About 50% of these risks can be explicated by poor school performance, when controlled for factors such as cognitive ability and birth parental characteristics (Forsman and Vinnerljung, 2012). Therefore, children in public care are not only vulnerable as children, their childhood situation can lead to lifelong struggle. Hence, society should ensure that these children are provided with resources to counteract their exposed situation. A number of different interventions have been designed for this reason, aiming at improving literacy, often numeracy, and in some cases attendance (Forsman and Vinnerljung, 2012). The majority of these interventions are rather extensive since they are designed as one-to-one tutoring. Most of these intense interventions have shown good results regarding reading, but most of them cannot provide significant results regarding mathematics.

In an attempt to design a low-cost intervention aiming at enhancing literacy and numeracy for children in public care, Rose Griffiths took initiative to enrol this group of children in a club, Letterbox Club, (LBC), which she presented at SEMT six years ago (Griffiths, 2013). In short, membership in LBC is offered through the social services and those who become LBC members receive a parcel each month for six months. The parcels includes books, mathematical games and

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stationary items. LBC in England started 2003 with 20 children, 8–11 years old, and now more than 11,500 children in the UK, ages 3–13, are members. The LBC has spread to a number of countries, among them Sweden. In this paper we report from a pilot study of the Swedish LBC and its effects on mathematics. However, first we describe what is known of the impact of LBC from earlier studies.

Background

In a review from 2012 (Forsman and Vinnerljung, 2012), three successful interventions for children in public care with regard to mathematics was found; two one-to-one tutoring programs and the Letterbox Club. A Canadian intervention program used foster parents as tutors and they reached significant positive results in “mathematical computations through counting, identifying numbers, solving simple oral problems, and calculating written math problems” (Flynn, Marquis, Paquet, Peeke and Aubry, 2012). However, these foster parents and their foster children had to be chosen by the social workers as eligible, which meant that the foster parents were assessed to read well enough through a test, and willing to undergo instruction of how to perform the tutoring. The foster children had to be assessed as likely to benefit from the program by their welfare office and to be fluent in English.

The tutoring went on for a minimum of 3 hours each week in 30 weeks. Under such circumstances it is not surprising that it had good effects. The other tutoring program, that Forsman and Vinnerljung (2012) reviewed, involved volunteering teachers who tutored foster children for approximately 33 hours altogether during 20 weeks. This study showed small, but significant effect on the mathematics achievement. Forsman and Vinnerljung conclude that interventions for academic improvement regarding foster children is not well researched and intervention studies tend to have weak design. A search in March 2019 through three major electronic data bases revealed no other successful intervention for children in foster care with documented significant improvement of mathematical skills.

Letterbox Club, (LBC), was acknowledged as successful for one of the two years that were evaluated (Forsman and Vinnerljung, 2012). This evaluation showed that the LBC members in 2007 performed significantly better than expected in mathematics (Griffiths, Comber and Dymoke, 2010). The assessment of mathematics was conducted through a specially designed instrument measuring counting, place value and mental arithmetic, which reflected the mathematical content of the games in LBC. The test results were transferred to match the national curriculum levels. Hence, each child was tested in pre- and post-tests and the results were compared to national levels of expected outcome of schooling, thus comparing the child with the expectance of his or her cohort. One might argue that studies of the effects might be more objective if they are run by independent researchers, that is researchers not involved in the development of the intervention. These positive results for the children’s mathematic knowledge (and
reading skills) was confirmed by a such an independent study (Winter, Connolly, Bell and Ferguson, 2011). However, when the same research team conducted a randomised controlled trial investigating LBC members in Northern Ireland, they could not find any effect on the children’s reading skills (Mooney, Winter and Connolly, 2016). Their study did not include mathematics.

In addition to the mathematical assessment, literacy skills and affective values have been investigated in relation to LBC. Here we briefly review findings related to affective values, since the child’s view of the parcels might affect his or her willingness to play the games. Most studies report of positive responses to be a LBC member, both from children and foster parents (Griffiths, 2012; Griffiths and Comber, 2013; Griffiths et al., 2010; Hancock and Leslie, 2014). For example, there are reports that the content of the parcels supported the growth of relations between the foster child and foster parents (Griffiths et al., 2010; Hancock and Leslie, 2014) and the children became more active as learners (Griffiths and Comber, 2013). The ownership of books and games itself, as well as the joy of getting a parcel by post directly addressed to the child, is thought to be important factors in LBC (Griffiths, 2012). However, one study in Northern Ireland reports that even if this was true for some children, other children felt that the books and games burdened them and caused pressure that they had to work with the material (Roberts, Winter and Connolly, 2017).

As the review of studies concerning LBC show, there are few studies which have investigated participating students’ attainment in mathematics. Those we have found, namely two from Rose Griffiths and her colleagues (Griffiths, 2012; Griffiths et al., 2010), later confirmed by Winter et al. (2011), report of promising results in terms om LBC members keeping up in the same development pace as their peers, which they most often tend to lag behind from. In the following we report the mathematics part of a pilot study in Sweden, which have been reported before in Swedish in a different format by using a slightly different analysis (Andrews, Larsson and Ridderlind, 2017). Before reporting the study, a brief description of the Swedish LBC games is provided.

Mathematical games

The idea to use mathematical games to enhance mathematical knowledge and skills is strongly supported by studies in pre-school settings and have shown to have potential to bridge the gap between underprivileged children and privileged (Burger, 2015; Ramani and Siegler, 2008).

The mathematical games in the Swedish pilot study were of three types; LBC games from the UK translated into Swedish, commercial games such as Yatzy, and specially designed games for the Swedish LBC. Common for all games were that playing them, afforded potential to enhance number sense and arithmetic skills. The LBC games from the UK were all tested by a number of children in the
UK and the translation were made by the first and third author to ensure that it was both culturally correct and kept the mathematical idea. For commercial games we provided extra sets of rules to make it easier to start playing the games. The newly designed LBC games were influenced by results from the pre-test. In the pre-test the LBC members had shown to have more trouble than their peers to solve subtraction tasks, hence we wanted to include games that could support their subtraction skills. One such a game was a ‘snake and ladders’ game with two dice, one with the numbers 12–17 and the other numbered 6, 7, 8, 8, 9 and 9. Playing it would supposedly support children’s automatization of subtraction facts. Most of the newly designed games were tested by a panel of children before they were sent to the LBC members.

**Method**

In order to measure the children’s achievements without a control group of other children in similar circumstances we decided to compare them with their peers, thus testing both the target children and their classmates. To administrate such a testing we contacted the LBC members’ teachers and asked if they could assist by giving the test to their classes two times, once before the intervention and once after. Thus we did not only have a pre- and a post-tests for the LBC members but also for other children, who participated in the exact same schooling and hence mathematics instruction, but without receiving any mathematical games.

The Swedish LBC is not only enrolling children in foster care but also children living in families who receive long-term financial support from society. This group is extra vulnerable, hence they are protected by laws of secrecy concerning who is allowed to take part of information about them. Therefore, we describe ethical arrangements undertaken to protect their integrity.

**Ethical considerations**

The procedure to test the target children’s whole classes, allowed us to get access to many children’s test results, and in order to keep them anonymous we made an agreement with the social welfare offices responsible for each LBC member. The welfare offices administered the test booklets, masking all students’ names and replacing them with a code. The code made it possible to follow each LBC member between the tests. The peers’ tests were coded with the same code representing a certain class, in order to make it possible to keep each LBC member in relation to his or her classmates, but which ruled any possibility to follow other children than the LBC members. LBC members’ legal guardians had accepted participation of both the membership and the research. By Swedish law the teachers are not allowed to be informed whether a child in her class is living in a family on long-term welfare support, hence the social secretaries needed to be involved in coding the test booklets.
Participants

There were 77 children, 39 girls and 38 boys, who participated in the pilot LBC in 2015, 54 of them were in foster care and 23 lived in families who received economic support from the social welfare on a long term basis. The children were between 7 and 11 years old, the majority (71) were 8–10, which is the group we studied. Out of the 71 children 54 were included in the study, since their teachers accepted our invitation to participate. When it was time for the post-test, 6 classes did not carry it out due to LBC members had changed school or other, for us, unknown reasons. Therefore, we can report of 48 LBC members and their 875 peers. These were distributed on three age levels; 14 LBC members and 270 peers in grade 2–3, 15 and 223 in grade 3–4 and 19 and 382 respectively in grade 4–5.

Tests

The pre-tests were distributed in May and the post-tests in January the following year. The parcels were sent from June to December. This entails that the pre-tests were carried out in the end of grade 2 and the post-tests in the middle of grade 3. Similarly, the middle group took the pre-test in grade 3 and the post-test in grade 4, and the oldest group took the pre-test in grade 4 and the post-test in grade 5. From now on we write about the youngest group as grade 3, since they belonged to grade 3 for the main part of the study, consequently the middle group is called grade 4 and the oldest grade 5.

The tests for grade 3 and 5 were designed by using tasks from a well-used Swedish test instrument developed by the PRIM-group, which is the department responsible for the National test in mathematics in compulsory school. The tasks we used from the test focussed on numbers and arithmetic and were designed for grades 2 and 5 respectively, hence some adjustments on number size were made for grade 3 and 4, but basically the same type of tasks were used in both grades.

For grade 4, the national tests were used as a pre-test. Since the spring term is very test-intensive for students and teachers in grade 3, we could not burden them with yet another test. The post-test tasks were of the same mathematical structure as the tasks from the national test we used as base-line data, but slightly different since the test tasks are under secrecy. The grade 4 tasks had the same or similar structure as the tasks for grade 3 and 5.

There were four types of tasks in the tests. All tasks were structurally identical in all three grades, but number size differed. In subtraction all numbers were less than 100 in grade 3, while three-digit numbers were used in grades 4 and 5, all according to the curriculum. Here are the four types of tasks exemplified.

1: Tasks evaluating number facts, e.g. “Calculate12 − 5 = ___ “and “Fill in the number before and after ___, 39, ___”.

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2: Tasks evaluating the static aspect of the equal sign and understanding of operations, e.g. “Fill in the missing number $4 = \_ \_ \_ – 12$”.

3: Tasks evaluating numerical arithmetic and adequate use of method, e.g. “Calculate $34 – 18$. Show your work!”

4: Tasks evaluating problem solving, e.g. “Malte has 4 pencils, Elsa has 3 times as many pencils. How many pencils has Elsa? Show your work!”

**Evaluation of answers**

Tasks of type 1, and 2 were given 1 point for correct answer and 0 for wrong. Tasks of type 3 were rewarded 2 points for both adequate method and correct answer, 1 point for either correct answer or adequate method with minor calculation error and 0 points for inadequate method and incorrect answer as well as no answer. Type 4 tasks were given 3 points for both adequate method and correct answer, 2 points for correct answer without showing what method was used, 1 point for an attempt to solve the problem with an adequate method but incorrect answer, 0 points for inadequate method and wrong or no answer.

This somewhat unusual evaluation system is in line with the Swedish national tests. In calculation tasks students are generally rewarded one point for using an adequate method, or performing the method adequately even though minor calculation errors occur and the correct answer is rewarded with yet one more point. For example, if a student correctly uses a vertical algorithm and handles regrouping correctly, but writes that $6 + 7 = 14$, see Figure 1, he/she gets one point for using the method correctly even though the answer is wrong. On the same note, text problems are rewarded one point for interpretation into an adequate operation or method even though the rest is erroneous, one more point for a correct answer, which implies that the problem were correctly interpreted, and yet one more point for demonstrating an adequate method as well as the correct answer.

![Figure 1: Correct method, wrong answer](image)

**Analyses**

For each test, students’ scores for each task were recorded in a database and their total test score calculated. From this average scores were calculated for each group on each test item. This allowed us to compare LBC members’ scores on each task as well as the whole test with their peers by means of t-tests. The comparisons undertaken were as follows:
Pre-test: comparing LBC members with their peers
Post-test: comparing LBC members with their peers
LBC members: comparing pre-test scores with post-test scores
Their peers: comparing pre-test scores with post-test scores

In addition, the distribution of the students’ scores on each task were compared in several ways. This allowed us to see how LBC members compared with their peers and, importantly, how LBC members compared with themselves from one test to the other; was there evidence of more students getting higher marks on each task and, if so, which ones? Associated with each of these crosstabs, a chi square test was undertaken in this respect, with the same four comparisons as above.

**Results**

First the overall results for the pre- and post-test respectively are presented. Thereafter, separate tasks on which LBC members and their peers scored considerably different are offered.

**Pre-tests**

In grade 3 the LBC members achievement were 98.4 % of their peers in the pre-test, which is not a significant difference. But in grade 4 the difference between LBC members and their peers was significant as the LBC members only got 89.7 % of their peers. In grade 5 the pre-test results also displayed a difference, which was close to significant with a p-value of 0.063, as the LBC members achieved 91.8 % of their peers.

This demonstrates that there is a difference in achievement to the LBC members’ detriment in all three grades. The difference seems to grow from grade 3 to grade 4, where it became significant and in grade 5 it was close to significant but the difference between groups were slightly less than in grade 4.

**Post-tests**

In grade 3, the post-test demonstrated that the LBC members had enhanced their achievement more than their peers, but it was not a significant difference. The grade 4 LBC members, who performed significantly less than their peers on the pre-test, were still lagging behind, but the gap had decreased, now scoring 91.9 % of their peers, which is not a significant difference. In the same way the grade 5 LBC members had improved more than their peers, now achieving 96.3 % of their peers compared to 91.8 % on the pre-test. This difference is not near significant.

This demonstrates promising results that the LBC members seem to gain mathematically from the membership. However, the improvements are not significant so it is not possible to draw clear conclusions from the study.
**Task with noteworthy difference**

When examining the tasks we found a number of tasks where the difference between the LBC members and their peers stood out. Especially subtraction tasks seemed to result in significantly lower scores for the LBC members in the pre-test. A telling example is “Calculate 403 – 297. Show your work!” On this task the LBC members’ mean score was 0.53 while their peers scored 1.05 on average; that is LBC members only performing half as well as their peers in respect to points awarded to this task. This task and two others are presented in table 1, showing the results on pre- and post-tests for LBC members and peers respectively.

In the table we display an example of a task from grade 3 where the LBC members scored better than their peers on the post-test. The task was a two-step problem in which a picture shows a book, a football and a skipping rope together with price tags. The question was “Sam buys two books and a skipping rope. How much does he need to pay? Show your work!”. This and the subtraction task, 402 – 297, can be considered to show some extremes in which the LBC members improved much more than their peers. Therefore, we also show an example of a task in which the LBC members improved more moderately, a problem for grade 5 in which two pictures are shown. One with a child on a scale, where the scale shows 37 and on the other picture the same child is holding his dog while the scale shows 51. The question is “How much does the dog weigh? Show your work!”

<table>
<thead>
<tr>
<th></th>
<th>Pre-test</th>
<th>Post-test</th>
<th>Percentage improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Grade 3</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem solving</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(max 3p)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LBC</td>
<td>1.86</td>
<td>2.43</td>
<td>30.8 %</td>
</tr>
<tr>
<td>Peers</td>
<td>2.08</td>
<td>2.39</td>
<td>14.6 %</td>
</tr>
<tr>
<td>Percentage of peers</td>
<td>89.2%</td>
<td>101.8 %</td>
<td></td>
</tr>
<tr>
<td><strong>Grade 5</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>403 – 297 (max 2p)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LBC</td>
<td>0.53</td>
<td>1.05</td>
<td>100 %</td>
</tr>
<tr>
<td>Peers</td>
<td>1.01</td>
<td>1.29</td>
<td>28.4 %</td>
</tr>
<tr>
<td>Percentage of peers</td>
<td>52.4 %</td>
<td>81.5 %</td>
<td></td>
</tr>
<tr>
<td><strong>Grade 5</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>51 – 37 (max 3p)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LBC</td>
<td>1.42</td>
<td>1.89</td>
<td>33.3 %</td>
</tr>
<tr>
<td>Peers</td>
<td>2.01</td>
<td>2.37</td>
<td>17.5 %</td>
</tr>
<tr>
<td>Percentage of peers</td>
<td>70.6 %</td>
<td>80.1 %</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Results for three problems comparing LBC members with their peers between pre- and post-tests
Discussion

The results from this study are more than promising. Even though the over-all results do not show significant differences for LBC members compared to their peers, all three age-groups had developed mathematically more than their peers. The results of certain tasks have significant differences, which is interesting and sometimes intriguing. The LBC members demonstrated significant less knowledge on subtraction tasks in the pre-test. We have no explanation to that, but the fact that students in the ages 8–11 generally find subtraction hard seemed to effect these children especially much. To some of the tasks, not only subtraction tasks, the LBC members displayed a very strong development. The only difference, known to us, between the LBC members and their peers were the membership. Naturally, it is possible that these children participated in other structured activities such as extra teaching with or without special needs teachers. But that is also true for their peers.

One major weakness in this study is that we had few LBC members in each age group. With only 14 LBC members compared to 270 peers in grade 3, it is hard to find strong correlations with statistical methods. In the RCT design used by Mooney et al. (2016) they had about 60 students in each of the two groups, which is a much stronger design. They chose, however, to evaluate only literacy skills, which might be due to more well-recognised test for literary skills compared to mathematics. This lead us to discuss what types of test we need to assess students’ mathematical knowledge and what tests are available. In Sweden we have an abundance of test for screening students with special needs and for assessing against the lowest acceptable level. These tests probably have ceiling effects when used to assess development for a cohort. This is also a possible bias in this study, the students who performed well in the pre-test had no opportunity to demonstrate a possible development.

Finally we think this small study is encouraging and that it needs to be followed by more strictly designed studies. There are few studies concerning the development of mathematics through LBC, and so far there has not been any thoroughly designed randomised test study with a control group focussing this issue. The results from this small study in combination with earlier positive results (Griffiths, 2012; Griffiths et al., 2010; Winter et al., 2011) and the knowledge that playing games can enhance mathematical skills with potential to bridge the gap between underprivileged children and their peers (Burger, 2015; Ramani and Siegler, 2008), is enough to go further and study the possible correlation of LBC and better mathematical achievement. Even if only a few children at risk are given better possibilities to succeed in school, it is a low-cost intervention and these few children are worth the efforts.
References


FUNCTIONAL THINKING OF THIRD GRADE STUDENTS: A STUDY FROM EARLY ALGEBRA FRAMEWORK

Esperanza López Centella

Abstract

This work explores how students of third grade of primary education (8-9 years) think about questions based on a generalising problem, involving a functional relation pictorially presented. We examine the strategies, functional relations and rules, and systems of representations employed by students through the analysis of their responses to paper-based questionnaires. Our findings illustrate the variety and combined use of strategies that they employ to express the relationship between co-varying quantities; the importance of numbers size involved in questions to foster functional thinking, and the influence of the picture used in the problem statement for their understanding and solving processes. As an outcome of the study, teaching implications on the design of tasks to promote functional thinking in early grades are outlined.

Keywords: functional thinking, strategies, representation systems, primary education, early algebra

Theoretical framework and background

Early algebra, emerged in the 80’s in the United States of America, can be understood as a particular approach to the teaching and early learning of mathematics. Among its main foundations, it is highlighted the importance of: (a) begin early (in part, by building on students’ informal knowledge); (b) integrate the learning of algebra with the learning of other subject matter (by extending and applying mathematical knowledge); (c) include the several different forms of algebraic thinking; (d) build on students' naturally occurring linguistic and cognitive powers (encouraging them at the same time to reflect on what they learn and to articulate what they know); (e) encourage active learning (and the construction of relationships) that puts a premium on sense-making and understanding. (Kaput, 1999, p. 3.)

It is not an aim of early algebra to advance to the infantile and primary education the algebra part of the mathematical curriculum – currently conceived in later educational stages – but to initiate the treatment of proper algebraic notions in a meaningful and accessible way for schoolchildren (Carraher, Schliemann and Schwartz, 2008). Prior to the algebraic language, abstract thinking finds a variety of channels and symbolisms to be manifested by children. One of the main objectives of early algebra is to favor the transition from numbers, understood as particular entities, to numerical patterns and regularities by means of an abstraction exercise. This includes generalizing arithmetic operations as functions (Castro, Castro and Rico, 1995). With these purposes it was proposed the so-called

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*Functional thinking* (Smith, 2008). Functional thinking is understood as “representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances” (Smith, 2008, p. 143). There are many research works reporting the capacity of schoolchildren to identify functional relations (Blanton and Kaput, 2004, 2011), to use systems of representations to express them (Brizuela and Earnest, 2008), and to employ methods of generalization in problems involving linear and non-linear functions (Amit and Neria, 2008).

According to NCTM Principles and Standards (2000, p. 37), students in the elementary grades should be able to: (1) understand patterns, relations, and functions; (2) represent and analyse mathematical situations and structures using algebraic symbols; (3) use mathematical models to represent and understand quantitative relationships; and (4) analyse change in various contexts. In relation to this, questions on the types and design suitability of tasks to work and promote functional thinking arise (Stacey, 1989). Generalising problems – as they were called by Lee and Wheeler (1987) – are a type of problems which can be solved by examining special cases, organising the results systematically, finding a pattern and using it to get answers. We are interested in investigating how students spontaneously tackle them, without having been instructed on it at classroom. We consider that knowing the ins and outs of their natural ways of thinking when dealing with these problems will shed light about those relevant aspects for the optimization of the design of tasks aimed to stimulate functional thinking in early grades students. In addition, it could be inspiring for teaching techniques focused on activities with an algebraic character at this stage.

In our study, we consider three modes of analysing patterns and relationships, as a framework to discuss the kinds of functional thinking found when analysing the collected data: (1) recursive patterning or recurrence, which involves finding variation within a sequence of values; (2) covariational thinking, based on analyzing how two quantities vary simultaneously and keeping that change as an explicit, dynamic part of a function’s description (e.g., “as x increases by one, y increases by three”); and (3) correspondence relationship, based on identifying a correlation between variables (e.g., “y is 3 times x plus 2”) (Smith, 2008).

**Objectives of the research**

The present work is part of a larger research project aimed to explore, from a qualitative approach, the functional thinking of primary education students. In the context of a contextualized generalising problem that implicitly involves the function \( f(x)=3x+2 \), pictorially presented to third graders of Primary Education, we establish the following specific goals: (1) identify and describe the solving strategies and systems of representation employed by the students; (2) recognize and describe the relations between variables (correspondence, covariation,
recurrence) and functional rules identified by them; (3) detect relevant aspects for the design of tasks aimed to promote the functional thinking of third graders.

**Methodology**

We perform a qualitative, descriptive and exploratory research (Erickson, 1986) with a case study design (Yin, 1989) with eleven third-grade students (8-9 years old). It is a performance of several work sessions of 90 minutes, carried out during the second term of the three of a Spanish school year. The tasks applied along the study required near generalization (the order of the terms allows the use of strategies like making a drawing or using a recursive method) and far generalization (for which recursive methods turn out to be not adequate and finding a rule becomes necessary). This paper reports results from the second session, being the first one devoted to the work on a generalising problem presented through a picture. Figure 1 presents the statement of the task.

After a first review of the data and related research literature (e.g. Morales et al., 2018), we establish the different categories of analysis of strategies and systems of representation relying on the Grounded Theory (Corbin and Strauss, 1990). We consider the following strategies: (1) recursion, if the response of a question is obtained by merely considering the response given in a prior one; (2) operational, based on the performance of arithmetic operations; (3) counting of drawings; (4) direct, if explanations are not provided on how the answers were obtained; (5) particularization, if a general question is answered by a particular case; (6) generalization, if a generalization of the involved functional relation is provided; (7) use of the unit as a reference, meaning in general the application of a proportionality based on the unit case.

We distinguish numerical, verbal and pictorial systems of representation. For this latter, we examine which elements (tables, guests) were drawn, the accurateness of them (shape) and the relation established between the elements.

![Figure 1: Statement of the task (Blanton, 2008)](image-url)
Subjects

Students participating in the study are a class group of 11 third graders (8-9 years) of primary education from a public urban school in Andalusia (Spain). In general, they possess the mathematical knowledge set out in the corresponding Spanish educative curriculum (MECD, 2014). They had not received any classroom instruction on functional relations, not having either prior experience with this kind of task. Most of them have a very well developed number sense, thanks to their work with ABN methods since kindergarten (Cerda et al., 2018). At the time of the study, they are able to perform quickly arithmetic operations with relatively large numbers (e.g. $48 \times 25$, $998 \times 7$) through mental calculus.

Development of the sessions

The development of each session is structured in four parts: (1) oral presentation of the generalising problem to the whole class group by the research team (with the help of representations on the blackboard, posters, cards, manipulatives, etc. conceived for this purpose); (2) student work in large group on particular cases of the problem with the guidance of the research team; (3) student individual work on the tasks worksheets (Figure 1), with the attention to their questions of the research team (willing to provide agreed supporting comments); (4) sharing on the work previously done and justification of responses. The instruments for the data collection consist of the individual task worksheets and one notebook, giving rise to written responses of the students and field notes of all facts considered relevant by the research team during the development of the sessions (including student oral interventions during their resolution).

Presentation and discussion of the results

First of all, it is important to mention that the students were not acquainted with the trapezium shape. Although they recognised the table shape included in the problem statement as a quadrilateral, they did not know which kind of quadrilateral it is (the teacher confirmed that they had not studied yet classification of quadrilaterals). Dealing with it represented a great difficulty for many of the students, especially regarding the representation and transformation of the shape for the pertinent joining. On the one hand, a number of students wrongly represented the tables as squares or rhomboids. This conditioned for them the number of guests per table, as they tended to draw the same number of guests in each side of the table, since the sides had similar lengths in their drawings. This was also observed in some cases even when they got to properly reproduced the trapezium shape of the table, depending on the length given to each side. On the other hand, the glide reflection – reflection composed with translation – (or, equivalently, rotation composed with translation) of the figure, needed to join tables as indicated in the problem, was hard for some of them.

In general, all the students employed systems of representation of verbal, pictorial and numerical types. In some cases, they drew the tables but, for unknown reason,
they did not draw the guests around them, in spite of being an illustrative part of the representation. Next, we present some examples of student performances in order to illustrate the functional thinking that they evidence. The students names have been replaced by labels (S1, S2, …) in order to protect their identities. The student S1 manifests covariational thinking in her answers to the first and second questions (although she fails in numbers):

1. “They are 10 because you add 4”;
2. “They are twenty three because with 4 tables they are 15 more 4 more 4 more they are 23”.

The student S7 also evidences covariational thinking, in this case based on a proportionality rule, as his answers to the first and second questions show:

1. “11 because one always adds 3”;
2. “22 because if three are 11 the double is 6 then they will be 22”.

In both cases the students apply, according to their rules, the identified change in the quantities of tables to the quantity of guests.

The student S6 answered as follows the question (1): “11 guests, because it goes three by three”, noticing that with each additional table, 3 more guests can sit. Drawing on near and preceding particular cases where the 2 guests at the non-parallel sides are included (for 1 table, 5 total guests; for 2 tables, 8 total guests; etc.), he got correct answers to the first questions by applying the recurring rule based on the sum of 3. For the far case (50 tables in question (4), without the possibility of directly applying his recursive rule (since he did not have any near and preceding particular case), he operated on the given number of tables. From his previous performance he inferred that the number of guests is calculated by multiplying the number of tables by 3, applying a multiplicative rule: “150 guests. 50×3 guests = 150 guests”. On the other hand, his observation in question (5) “at the top and the bottom there can be two chairs and at the top and the bottom there can be 1 chair”, helped him to provide correct answers to the first questions even when he drew non-trapezium-shaped tables.

Figure 2 shows the original responses provided by S5.
The verbal explanations provided by S5 translate to English as follows:

1. “Joining one more table and putting the guests”.
2. “Joining 8 tables and putting the guests”.
3. “I do not know it”.

This accounts for the strategy of counting drawings employed by her (her numerical responses coincide indeed with the number of represented guests in each case). Due to the strong dependence of her answers on her representations, the far particular case (question (5)) was out of reach.

Student S8 provides the following explanation to question (5): “Because at one table fit 5 guests thus at 8 tables 40. I do it multiplying.” He considered the unit case as a reference, what leads him to a multiplicative rule in all his responses.

Table 1, Table 2 and Table 3 collect and complete the information presented.

<table>
<thead>
<tr>
<th>Recursion</th>
<th>Operational</th>
<th>Drawings</th>
<th>Part.</th>
<th>Gen.</th>
<th>Direct</th>
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</table>

Note. Part.: Particularization, Gen.: Generalization.

Table 1: Strategies used by the students

<table>
<thead>
<tr>
<th>Involved rules</th>
<th>Functional relations</th>
</tr>
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</tbody>
</table>
Note. The notations $g$ and $t$ stand for numbers of guests and of tables respectively, understood as variables of a function. The notation $g_n$ (for any natural number $n$) stands for number of guests, understood as the $n$-term of a succession.

Table 2: Rules and functional relations identified and employed by the students

<table>
<thead>
<tr>
<th></th>
<th>Pictorial</th>
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<th>Verbal</th>
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<tbody>
<tr>
<td></td>
<td>Drawing of tables</td>
<td>Drawing of guests</td>
<td>Guests per table</td>
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Note. NA: Not applicable.

Table 3: Systems of representation employed by the students

Conclusions

Our results confirm that, from early ages, there are students able to deal with tasks that promote functional thinking. This complements the results obtained in other research works on functional thinking with students of Primary Education (Blanton and Kaput, 2004, 2011; Brizuela and Earnest, 2008; Morales et. al, 2018).

On functional relations identified and used by the students

The most frequently identified functional relations by those students who evidence functional thinking (10 out of the 11 participants in this case study) are correspondence and recurrence. This coincides with the findings of other studies (Stephens et al., 2012; Tanişli, 2011). We highlight that some students responded the questions through more than one functional relation: recurrence plus correspondence, and correspondence plus covariation. In general, none of these
combinations led to valid functional rules. The main sources of student mistakes were, on one side, not to consider the constant quantity (2) involved in the function that models the problem (forgetting/ignoring the two people that sit in the extreme sides of any length chain of the trapezoidal tables). On the other side, a relatively common fail was to apply a proportionality rule, taking the unit case as a reference. Even when it is not the focus of the present research, we consider that the analysis of inappropriate student responses can provide relevant information on their knowledge and the way in which they use it. It points out those ideas that must be emphasized in teaching for a good learning of elementary mathematics (avoiding conceptual gaps). The common use of a recurring rule by the students could be explained by the fact that the task implicitly includes three consecutive particular cases (1, 2 and 3). They tried to find a patterning uniquely based on the changes of one the quantities (number of guests). This limited their chances to generalize the functional relation. Other studies avoiding consecutive numbers in questions document absence of students identifying recurrence (e.g. Morales et al., 2018).

**On strategies employed by the students**

Remarkably, many students involve more than one strategy in their responses. We observed that, in general, they used operational strategies and counting of drawings. The students were really acquainted with calculus, so that perform arithmetic operations with multi-digit numbers was affordable for them. The fact that they combined operational strategies with counting of drawings seems to be more related to their need of a visual support to try to find out the relationship between the quantities in play. Three of them got to generalize the functional relation involved in the problem, providing a verbal formulation of it (“multiply by three and add two”). Nevertheless, the scope of this objective was importantly limited by the recurrence relation in which many of them based their responses, as also pointed out in other works (Moss and London McNab, 2011).

**On systems of representation used by the students**

In general, all the students employ systems of representation of verbal, pictorial and numerical types. Most of them prove to be able to explain under a verbal representation system the way in which they obtain their responses (although only the descriptions of three of them correspond to the functional relation in play). Regarding the pictorial representation system, definitely, their drawings of the tables really influenced their identification of the functional relation involved in the problem. They tended to represent as many guests as they apparently could fit on each side of the tables that they had drawn. This altered the quantities and decisively conditioned their perception of the relation between them. In addition, the transformations on the figures needed to an appropriate joining of them was an obstacle for some students. Related facts to this were reported by Warren (2008). Nevertheless, the observation of a student relating the numbers of guests
and their location ("Because above and at bottom there can be two chairs and above and at bottom there can be 1 chair"), helped him to correctly answer first questions even when he drew inappropriate representations of the tables. This informs about the different uses of the pictures by the students in their solving processes: some students literally based their responses on what they could see and find out through their drawings, while others used them as an informal representation to visualize what they conceptually already had in mind.

**Implications for the teaching of functional relations in Primary School**

Regarding the design of tasks aimed to develop functional thinking, the size of numbers has been revealed as a relevant aspect. This prompts students who are using recurrence to consider the search for another functional relation to cope with those questions involving large numbers. In addition, this search could bring them closer to generalization. On the other hand, in view of our results, also the shape of the figures illustrating the functional relation of the generalising problem becomes important. To this respect, we recommend to choose a simple and familiar shape for the students (Blanton et al., 2011), in order to favour the concentration of their efforts in the identification of the functional relation involved in the problem.

Finally, it deserves to be mentioned that no written attempts of checking their rules once obtained (in order to validate or invalidate them) by any of the students were observed. Being essential in any mathematical activity, the checking of statements seems not to be a usual practice for students. Through this research report, we also stress the importance of emphasizing this in the teaching of mathematics, in particular, of primary education.

**Acknowledgement:** The author is pleased to express her deep gratitude to the research group “FQM-193: Didáctica de la Matemática. Pensamiento numérico” (funded by Junta de Andalucía), from which she is a member, as well as the Director and Teaching Staff of the Andalusian School participant in the study for their collaboration.

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Stephens, A., Isler, I., Marum, T., Blanton, M., Knuth, E. and Gardiner, A. (2012). From recursive pattern to correspondence rule: Developing students’ abilities to engage in
EMBODIED LEARNING IN MATHEMATICS EDUCATION – FOSTERING CHILDREN’S REPEATING PATTERN COMPETENCIES BY USING WHOLE-BODY MOVEMENTS

Kathrin Nordemann and Thomas Rottmann

Abstract

Mathematical learning processes are substantially based on concrete actions. Usually those actions exclusively involve fine motor actions using manipulatives. However, from embodied cognition theory, approaches have a broader view in this context and take the different domains use of manipulatives, gestures and body movements into account.

This paper gives a brief introduction into the concept of embodiment in mathematics education and outline taxonomies of embodied learning. On this theoretical background an exploratory study with 19 German first grade students, which intends to foster children’s mathematical learning by whole-body movements, is presented. Repeating pattern competencies were chosen as the relevant content domain to focus on. In this pilot study, three different movement games dealing with a variety of activities concerning repeating patterns were developed and implemented during seven teaching units. Children’s learning progress was evaluated using a pre- and post-test design. The findings of the study indicate a positive development of children’s patterning competencies, particularly concerning extending repeating patterns and identifying the unit of repeat in a given pattern.

Keywords: embodiment, mathematics education, repeating patterns, exploratory study, whole-body movements

Introduction

Concrete actions are regarded as an essential foundation for the development of mental operations and children’s cognitive development in general (Bruner, 1966, 1973; Piaget, 1970). Usually, concrete actions in mathematics education are construed as fine motor activities, using concrete (didactical) material to deal with (McNeil and Uttal, 2009; Uttal, Scudder and De Loache, 1997). A comparatively small number of approaches involve a higher amount of bodily engagement and

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focus on physical activities as a different type of concrete actions (Bayer and Rottmann, 2018).

Physical activities in the sense of whole-body movements are considered to be an important element of embodied cognition, provided that the “bodily activities […] are closely related to the mathematical content” (Tran, Smith and Buschkuehl, 2017, p. 11). Tran et al. (2017) emphasize the close relationship between body’s interaction with the world and cognitive processes. Although the concept of embodied cognition is generally accepted, ideas for the practical application in mathematics education are still missing.

The exploratory study presented in this paper deals with the previously underrepresented domain of whole-body movements in mathematics education. Examples for movement games, which closely connect the movement to the mathematical learning content, were designed and tried out in a group of first graders. The learning development of the students during the teaching unit was evaluated using a pre- and post-test design. Given the particular importance of elaborated pattern competencies for further mathematical developments (Lüken and Kampmann, 2018; Warren and Cooper, 2006), repeating patterns were chosen as the relevant mathematical content for all movement games.

This pilot study aims to develop, implement, and evaluate movement games dealing with repeating patterns, and thereby intents to build the foundation for further research on the relevance of whole-body movements for mathematical learning. Therefore, the main research interest is to analyse, to what extent children’s repeating pattern competencies can be fostered by physical activity in the sense of whole-body movements.

The Relevance of Concrete Actions for Mathematical Learning

In mathematics education the role of concrete actions in learning mathematical concepts is intensively discussed and strongly influenced by the theory of Piaget (1970) and Bruner (1966; 1973). Piaget (1970) describes different stages of children’s cognitive development. Thereby, cognition is initially directly linked to concrete actions, often performed with manipulatives. These activities build the foundation for the development of abstract thinking. According to Bruner (1966), cognitive development runs along three different levels: “from (a) acting on concrete objects to (b) forming images of the concrete constructions to (c) adopting symbolic notations” (McNeil and Uttal, 2009, p. 138). Both Piaget and Bruner highlight the relevance of concrete actions. Thereby, Piaget even was “an early proponent that sensorimotor activity aids in constructing knowledge and that bodily actions are not separate from, nor solely downstream, from the mind” (Johnson-Glenberg, Birchfield, Tolentino and Koziupa, 2014, p. 88).

While dealing with manipulatives extensively involves fine motor skills, different concrete actions with a higher level of motoric engagement are conceivable and are a subject of interest in embodied cognition approaches.
Embodiment and Embodied Cognition

The fundamental relevance of embodiment for cognitive processes is broadly shared both in general academics and in mathematics learning (Lakoff and Nuñez 2000; Tran et al. 2017). Accordingly, Askew, Abdulhamid and Mathews (2014) stress “that cognitive understandings in general and mathematics in particular are embodied – rooted in perceptual and physical interactions between the body and the world” (Askew et al., 2014, p. 65).

In general, embodiment is about the idea that cognition is not only formed by human’s brain, but also by other aspects of the body. Wilson (2002) points out that cognitive processes have “deep roots in sensorimotor processing” (Wilson, 2002, p. 625). This means that cognition is embedded in sensory-motor experiences of the body and its interaction with the world.

To support mathematical thinking in the context of embodiment, Tran et al. (2017) focus on the three domains use of manipulatives, gestures, and body movements as foundational for embodied mathematical cognition. Within the domain of whole-body movements two different forms can be distinguished: either the movements accompany the learning process only in time and not in content, or the movements “are closely related to the mathematical content with a negligible fitness component” (Tran et al., 2017, p. 11). The first approach refers to “Learning in Physical Activity”, whereas the second can be labelled as “Learning through Physical Activity” (Bayer and Rottmann, 2018, p. 67). In the latter form, the close connection between the movement and the learning content itself enables the sensory-motor information reception. Tran et al. (2017, p. 11) consider this form of whole-body movements as an important element of embodied cognition.

Taxonomies of Embodied Learning

As there are various research disciplines concerned with the notion of embodiment, different taxonomies of embodiment exist. Most taxonomies include the use of digital technologies, as the frequently received taxonomy from Johnson-Glenberg et al. (2014) does. These authors created an embodied learning taxonomy, taking into account the three different components amount of motoric engagement, gestural congruency and perception of immersion. Amount of motoric engagement describes whether children move on with their whole body or whether they move their hands just for gesturing. If there is a connection between the movements and the learning content, it is represented in the component of gestural congruency. The component perception of immersion is associated with the use of digital technologies and “is understood as it pertains to virtual reality and related technologies” (Johnson-Glenberg et al., 2014, p. 90; Skulmowski and Rey, 2018, p. 3). The combination of these components leads to four increasing degrees of embodiment. Whereas the first degree is a non-interactive seated learning setting without gestural congruency, the fourth degree
is realized by using whole-body movements and locomotion with gestural congruency (Johnson-Glenberg et al., 2014, p. 90).

This taxonomy has recently been criticized (Skulmowski and Rey, 2018; Tran et al., 2017). Particularly, the component perception of immersion is considered critical, because “[t]here is no guarantee, however, that emerging technologies will cue the kinds of body movements that have shown promise for effectively teaching mathematics” (Tran et al., 2017, p. 4). As a consequence, Skulmowski and Rey (2018) enhanced this taxonomy. The main aspects in their revised taxonomy are bodily engagement (instead of amount of motoric engagement) and task integration. Instead of focusing on the usage of digital technologies, there is a clear orientation towards the bodily activity itself. On the one hand, the dimension of task integration is defined by integrated and incidental forms of embodiment. Whereas integrated forms of embodiment can be defined as movements “that are closely related to the mathematical content” (Tran et al., 2017, p. 11), incidental forms of embodiment can be defined as movements “for the sake of fitness or motivation” (Skulmowski and Rey, 2018, p. 4; Tran et al., 2017, p. 13). On the other hand, the dimension of bodily engagement constitutes seated activities on a lower level (first and second degree of the taxonomy by Johnson-Glenberg et al., 2014) and performance of bodily movement on a higher level (third and fourth degree of the taxonomy). Skulmowski and Rey (2018) mention, that this dimension “and related notions of motor activation have been proposed as major characteristics of (educational) embodiment research” (Skulmowski and Rey, 2018, p. 5). Due to this fact, bodily movements in embodied learning settings without using digital technologies are the main interest of this paper. Accordingly, this paper only deals with those embodied learning settings that can be located on a higher level of bodily engagement (in accordance with Skulmowski and Rey, 2018), and in which the movements are integrated in the learning tasks.

Studies on embodied learning focusing on the effectiveness of whole-body movements, particularly in mathematics, are rare and largely missing. Among the few exceptions is the exploratory study by Bayer and Rottmann (2018).

Repeating Pattern Competencies in Mathematics Education

In accordance with Lüken and Kampmann (2018) a mathematical pattern can be defined as any predictable regularity. Three different types of patterns can be distinguished: spatial structure patterns, repeating patterns, and growing patterns (Lüken and Kampmann, 2018, p. 55). Particularly repeating patterns with a cyclic structure, consisting of a sequence of elements (the unit of repeat) that is repeated indefinitely (e.g., \(ABAB\ldots\), \(ABCABC\ldots\)), are frequently handled in the primary grades.

Dealing with repeating patterns develops general mathematical concepts and can also be regarded as an essential conceptual stepping stone in children’s
mathematical development. A longitudinal study reported by Lüken, Peter-Koop, and Kollhoff (2014) reveal a significant effect of children’s repeating patterning abilities one year prior to school on their mathematical competencies at the end of grade 1. Furthermore, a focus on pattern and structure during regular mathematics education shows significant effects on first graders’ competencies in arithmetic (Lüken and Kampmann, 2018, p. 64). Thereby, understanding the structure of a repeating pattern and identifying the unit of repeat are the most relevant aspects regarding other mathematical domains like subitizing, partitioning, counting or multiplicative reasoning (Mulligan and Mitchelmore, 2009). In addition, elaborated repeating patterning abilities in general are also relevant for further mathematical development, e.g. for functional thinking and algebra (Warren and Cooper, 2006).

Tasks and interventions in previous studies are frequently based on a variety of activities dealing with concrete materials to foster children’s repeating pattern competencies. The most common activities are the ones by Warren and Cooper (2006), including reproducing, extending and interpolating, i.e. filling in missing elements, repeating patterns as well as identifying the unit of repeat. The complexity of a pattern and, in connection with this, children’s success rate in dealing with patterns is strongly influenced by the number of elements of the unit of repeat and by the type of activity conducted. For example, a unit of repeat consisting out of two elements ($AB$) is less complex than one consisting out of three elements ($ABC$). In addition, several studies show evidence that reproducing or interpolating a repeating pattern is much easier than extending a repeating pattern (Clarke, Clarke, Grüßing and Peter-Koop, 2008, p. 269; Lüken, 2012, p. 98). The intervention in the present exploratory study picks up some of these activities by using whole-body movements instead of concrete materials to foster children’s repeating pattern competencies. Accordingly, children have to perform a sequence of single whole-body movements (corresponding to the unit of repeat) to reproduce or extend a repeating pattern. The competence to identify the unit of repeat is fostered by the accompanying verbal reflections of the whole-body movements.

**Exploratory Study**

To gain insight into the potential effectiveness of using whole-body movements to foster children’s repeating pattern competencies, we developed movement games with a close connection to that mathematical content. Therefore, specific movement games dealing with a variety of activities relating to repeating patterns have been developed and tested in an exploratory study in a German primary school with 19 first grade students. Four of these students were classified as experiencing difficulties in mathematics learning by their mathematics teacher, based on their current performance in the field of arithmetic.

The survey included seven teaching units, consisting of conducting a diagnostic tool in a pre- and post-test design as well as of carrying out the movement games
in a gymnastic hall. Both the diagnostic tool and the lessons were conducted by the first author. Apart from these units, there was no teaching of repeating pattern in the regular lessons.

The main research question was: To what extent can children’s repeating pattern competencies be fostered by whole-body movements? Furthermore, one special focus is on the learning progress of children with difficulties in learning mathematics.

**Methodology**

The exploratory study took place in September and October in 2017. Participants were 19 German first grade students (Mage = 6 years and 5 months). To identify children’s repeating pattern competencies a diagnostic tool in form of a paper-pencil test in a pre- and post-test design was conducted in the first and in the last lesson of the intervention. The test contained 16 different items to reproduce (3 items) and extend (5 items) repeating patterns with various units of repeat as well as tasks demanding to identify the unit of repeat (4 items) and to interpolate a pattern (4 items; Figure 1). In total, five different types of units of repeat were used (AB, ABB, ABC, ABBC, ABCC). To ensure better comparability of results, both pre- and post-test contained identical tasks concerning the repeating pattern activities.

For the intervention, three typical movement games from physical education (e.g. “Reversal Relay Race”) were adapted. The leading principle in the adaptation was to closely relate the movement to the mathematical content, in this case, to the repeating pattern’s structure: the unit of repeat and its repetition. For this purpose
the different repeating pattern activities were used. In the remaining five lessons of the intervention these movement games were implemented. A reflection phase to direct children’s attention to the unit of repeat in the pattern was integrated into each game.

**Example of a Movement Game: Reversal Relay Race**

A Reversal Relay Race is a racing competition. The group is divided into several teams. Each member of a team has to cover the distance between the starting and the reversal point and then runs back to the starting point. The next team member starts the racing competition.

In this Reversal Relay Race two of the common repeating pattern activities, *extending a repeating pattern* and *identifying the unit of repeat*, were used to foster children’s repeating pattern competencies. For example, the children extend a repeating pattern by rerunning a different unit of repeat consisting out of single whole-body movements like *jumping with one’s legs apart* – *jumping with one’s legs tightly* (AB) or like *jumping with one’s leg apart twice* – *jumping with one’s legs tightly once* (AAB). Also children mention that there are different kinds of units of repeat. During the explanation phase of the game there was a linguistic reflection with conscious phrases like: “First, you jump with your legs apart, second you jump with your legs tightly. And again: first..., second...” Furthermore, question like “Can you identify the movements you repeat?” were used to direct children’s attention to the unit of repeat while performing the whole-body movements.

**Results**

Due to the exploratory design of the study, the results are only generalizable to a limited extend. Nevertheless, the findings can be regarded as first indicators for the effectiveness of using whole-body movements in developing patterning abilities.

The results of the pre- and post-test show a positive development of children’s repeating pattern competencies. Whereas in the pre-test, an average of 9.2 tasks (from 16 tasks in total; 58% of tasks) were answered correctly, the number of successfully solved tasks increased to an average of 12.7 tasks (79% of tasks) in the post-test. Particularly in the domains *extending a repeating pattern* and *identifying the unit of repeat* there was a high increase in the number of correct answers in the post-test compared to children’s solution in the pre-test (Figure 2). Furthermore, it is conspicuous that there are considerable differences within the sample regarding students’ mathematical competence level. Figure 3 illustrates the students’ intra-individual development in terms of the difference in the number of correctly solved tasks between pre- and post-test. In general, there is a wide range in students’ development, from solving two tasks less up to 12 tasks more with correct results in the post-test compared to the pre-test. A striking result is that only two students show lower success rates in the post-test, whereas 12
students were able to improve during the intervention. Compared to the pre-test, the group of four students with difficulties in learning mathematics demonstrated an average increase of 5.25 more tasks solved correctly in the post-test. Student’s without difficulties in learning mathematics solely demonstrated an average increase of 3 more tasks solved correctly in the post-test. In general, no “ceiling effect” could be determined in the whole sample.

![Figure 2: Percentage of correct solutions in the pre- and post-test, taking into account different repeating pattern activities (N=19)](image)

Even if these findings are not generalizable, the results of this exploratory study indicate a positive influence of physical activity in terms of whole-body movements on children’s repeating pattern competencies. However, the design of the study cannot clarify to what extent the effects are caused by the physical
activity itself, or to what extent the positive development was influenced by confounding factors. In particular, the (linguistic) reflection of the activity encouraged in the reflection phases following each movement game could have played a special role. The verbal description of the movement, emphasizing the unit of repeat in each pattern, could have had a strong impact on children’s development. This is consistent with the view, that the description and reflection of an activity is essential to build “an anticipated (invariant) relationship between an activity and its effect” (Tzur, 2007, p. 275).

Outlook and Implications for Further Teaching Studies

The movement games proved suitable for fostering first graders’ repeating pattern competencies, by offering appropriate learning opportunities for all students, independent of their individual competence level. However, there is still an obvious need for further development and research in the domain of embodied learning in mathematics education. Therefore, this exploratory study serves as the basis for a subsequent research project by the first author that is currently in preparation.

An important consequence of the exploratory study is to modify the design of the subsequent study. To get more meaningful results, a larger teaching study should include control groups in order to determine the effect of embodied learning settings in contrast to settings using solely manipulatives or iconic representations. It is also important to analyze the impact of (linguistic) reflection phases accompanying the physical activity.

Regardless of the design of the study, it is worth considering to choose a different mathematical topic and to analyse if there are similar effects to the domain of pattern abilities. A focus on arithmetical competencies, e.g. in developing appropriate basic ideas for arithmetic operations, seems promising. In this context, initial results of the exploratory study by Bayer and Rottmann (2018) point to positive effects of movement games to develop basic ideas of multiplication.

References


REFLECTION OF VIDEORECORDINGS AS A PART OF THE CREATION PROCESS OF PROSPECTIVE KINDERGARTEN TEACHERS’ PROFESSIONAL PORTFOLIO

Eva Nováková

Abstract

The contribution gives information about a research aimed at construction of one component of professional portfolio of prospective kindergarten teachers by means of reflective teaching. During the stages of reflection and subsequent analysis aiming at the development of mathematical pre-literacy, the video-based observation method was used. We present an analysis of an activity aiming at the development of geometrical concepts. We use the ALACT reflection model for processing materials to be included in the portfolio. The research outcomes show that the materials focusing on various aspects of children’s mathematical pre-literacy development in the real-life kindergarten environments can be used for the construction of prospective kindergarten teacher portfolios.

Keywords: Kindergarten teacher, reflection, professional portfolio, video recording, ALACT model.

Introduction

A professional kindergarten teacher preparation has a significantly multiple field character. Teacher’s competences are focused on professional knowledge and skills, communication, educational process and both reflection and self-reflection (Syslová and Chaloupková, 2015). The role of mathematical component in prospective kindergarten teachers’ education is marginal. Based on previous research (Nováková and Novák, 2019), the prospective kindergarten teachers’ relationship to mathematics is various, but not completely negative. They usually realize the necessity of mathematical skills for real life, which can be positively reflected in their professional attitude to children’s pre-literacy development.

In our contribution, we aim to discuss specific activities which we analysed during our research and subsequently used it for creating one component of professional portfolio of prospective kindergarten teachers. In this, we focused on the development of geometrical ideas and concepts, which is a part of children’s pre-literacy which we believe is important for their personal development. Prospective teachers can create this component also with the help of other parts of their knowledge acquired in their previous mathematical training, such as set theory, elementary arithmetic or geometry. The portfolio can facilitate the professional development of the teacher only when it is based on the process of collecting, selecting and reflecting. Korthagen and Vasalose (2005) mention that the portfolio enables to focus not only on reflection of professional knowledge, i.e. theory, and on “observable” aspects of the profession and its performance (skills, specific

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ways of handling certain situations) but also on a “deeper insight” into one’s professional identity and its formation, one’s beliefs and attitudes.

**Theoretical background**

Preparatory university education is characterised by searching for ways of changing the academic approach based solely on theoretical knowledge. In most highly developed countries, the dominant teacher model has become a reflective teacher practitioner (Schön, 1987; Wubbels and Korthagen, 1990). The reflective practice model as a specific or clinical understanding of professional education is usually applied in the context of teaching and learning constructivism culture where the knowledge acquisition is usually related to practical experience reflection (Janík et al. 2013; Pihlaja and Holst, 2013). In the process of professional development of prospective kindergarten teachers we follow the realistic attitude to teachers’ education (Korthagen et al., 2011).

Professional portfolio creation is considered an essential part of reflective education (Syslová et al., 2018). In the last decades, the issue of portfolio in education, both in general and in the particular case of teacher training, has become a world-wide phenomenon. Our understanding of the concept as well as ways of its implementation into educational practice has various forms. Dysthe and Engelsen (2011) show the diversity of models and ways of portfolio implementation at national and international levels. The concept of portfolio is considered a flexible and authentic tool for development and evaluation of professional competence (Darling-Hammond and Snyder, 2000). Painter (2001) says that a portfolio helps to create an individualised teacher portrait, i.e. a professional reflecting on his/her philosophy and practical experience. The portfolio also helps to “realize the significance of the learning process and to acquire the ability to record independently his/her progress in knowledge, skills and attitudes to teacher preparation” (Tomková, 2018, p. 121). The teachers’ education involves a developmental or process portfolio which records student’s professional development, focusing on “Who am I becoming and how?” rather than “Who am I?” (Wyatt and Looper, 1999, p. 14).

Only when starting to work as a teacher does the novice teacher realize the difficulty and the complex character of a kindergarten teacher, which some authors call “a professional or reality shock” (Keiny and Dreyfus, 1989).

The use of portfolio is linked both to the professional development (Darling-Hammond et al., 2012) and to teacher evaluation (Campbell et al., 2004; Klecker, 2000). Dinham and Scott (2003) point out also the role of portfolio directly in the educational process, i.e. “in class”. They suggest that portfolio is an excellent feedback tool: for students on their progress, for teachers on the curriculum and the educational process. The authors suggest that the use of portfolio encourages student cooperation, critical view and evaluation of both themselves and others, as well as development of judgement skills and strengthens their internal
motivation. This is true especially for training of teachers of languages (especially English), mathematics but also arts and music and for pre-school prospective teachers and primary educations (Syslová et al., 2018; Slezáková et al., 2017). However, a reasonable process of portfolio construction is still a great challenge for prospective teacher educators (Stolle et al., 2005).

**Methodology**

**General background and sample**

In our research we use the method of a supervisor attendance video recording as a tool for applying a reflective teaching leading to a professional portfolio creation. We agree with Janík et. al. (2013) who claim that the value added to a video lies in the possibility of recording an authentic teaching situation in its complexity, which enables further comments and discussions. Video recording analysis is an essential part of a reflective teaching practice and it can and should help to develop professional teacher competences. In the new millennium many researches have been realized through video recordings, as Najvar (2011) says, e.g. lessons in mathematics recordings TIMSS (Hiebert et al. 1999).

Prospective teachers saw a video recording with an actual education activity realized by an experienced teacher. The activity took place in real settings of a kindergarten in a group of eight pre-school children, their age ranging from five to six years of age. The activity was aimed at practising a skill to create a construction made of colourful toy boxes based on following instructions.

The video was reflected in teaching. The aim of this joint detailed analysis was especially the urge of prospective teachers to “see” and “name” connotations and links of respective activities to development of mathematical concepts. This gives the prospective teachers a “model” for reflection of their own video recordings. Based on the reflected video recording a task was assigned for them. The prospective teachers were asked to perform a specific teaching activity in a given kindergarten and record it on video. After analysing this video recording, the activities could be used to create a developmental portfolio which documents their professional development.

Our research was realized in autumn 2018 in a group of 12 students of Faculty of Education of the Masaryk University in Brno. It was agreed with experienced kindergarten teachers in the region of Brno. All participants were female, students of third year of the attended form of study of Teacher training for kindergartens. We give details of one of the participants.

Anna is in her third year of studies and she graduated from a grammar school. She has a positive attitude to children, which highly influenced her choice of her prospective job of a kindergarten teacher. Her main motivation was her interest in working with pre-school children, and her own experience with working with children so far. Her relationship to mathematics can be described as following:
“At the elementary school I enjoyed more the lessons of Czech language, but when I got to the secondary school, I started to like mathematics more and more. I liked how individual parts of mathematics are joined and linked together and you need to understand the previous knowledge in order to master the following.”

In her opinion, analysis and reflection of early numeracy developmental activities recordings help to create a professional portfolio: “The reflections give me a battery of activities for developing children’s early numeracy skills.”

**Instrument and procedures**

Following a survey on reflection models by Píšová (2007), we chose to use the ALACT model (Korthagen et al., 2011). An important feature of this model is the overlapping of the process of reflection and the process of experimental learning. The acronym ALACT represents initial letters of its five stages:

1. **Action**: specific experience by means of an action. Prospective teachers perform a prepared activity with children. They can also make use of their experience obtained by means of reality observed at a kindergarten or through a video recording in class.

2. **Looking back**: reflection in an action and its link to a subsequent reflection on an action. The teacher “plays back” the events. At this stage it is necessary to develop descriptive language, i.e. terminology and professional perspective.

3. **Awareness**: a simple description of an observed reality is not enough for being aware of important aspects of education. This stage builds on mental operations such as analysis and evaluation, abstraction and conceptualisation leading to being aware.

4. **Creation of alternative procedures**: suggestions of possible alterations leading to better results, discussions on alterations.

5. **Trial**: an innovated trial based on reflection on action lesson (Janík et al., 2013, p. 192; see Korthagen et al. 2011, p. 74).

Further on we describe the stages done by Anna. However, ALACT analysis was used for all other research participants.

**Research results**

**ALACT model application**

1) Anna realized the activity in the group of 10 pre-school children. The children were divided into pairs. We called this activity “Designers and builders”. One of the children (the designer) was supposed to choose several Magformer shapes and show them to his/her mate (builder). The other child then took the same number of identical shapes. They sat back to back, not to see what the other one was doing. The designer made a construction using his/her selected shapes and described it orally to his/her mate, whose task was to build exactly the same construction based on the detailed oral description. Subsequently, the children swapped. The activity
task is meaningful and it offers an opportunity for creativity. The designer was to decide on the number, shape and colour of the shapes and the type of construction to be made.

2) When “looking back” on the video, Anna noticed if and how the child understood the assignment, if the child asked for help, how he/she dealt with problems and how he/she solved problems with the shape selection.

3) When analysing and evaluating, we suppose that the task aim is to teach children how to understand a particular activity. It is not important just to do something or to train a procedure. What is more important is the ability to make oneself understood and get one’s ideas through to others, to explain them and to give reasons for one’s actions. Last but not least, the activity should be interesting and motivating for the children. The task’s aim is to develop creative and pre-logical thinking, altogether with mathematical pre-literacy.

4) The fourth stage of the ALACT model seeks alternative procedures (creation of alternative procedures). Having analysed the video recording Anna suggested an alteration how to change the activity and realize it in a different way. Anna decided for an alternative task which she realized in her subsequent visit to the kindergarten, in the same class with the same group of children. It was obvious that the original task’s difficulty was inappropriate. The alternative activity thus reflected the need for simplification and specification of the task. The innovation was that the designer would work by a given 2D coloured model and would use planar shapes only, arranging them on the carpet.

5) The alternation was carefully thought through and designed. Anna tried it in real conditions (trial stage). The alteration activity stressed even more the need for detailed description of individual shapes used by the designer (their form, size and colour). The builders did not see them and they had to follow precisely the designer’s description. The children were more successful this time. The building of the construction aroused the desire for more constructions and for swapping the roles.

**Research summary**

Apparently, the child enjoyed the construction creation, but the task itself was very difficult for him/her. During the activity it became clear that the designers had difficulties deciding on the number and selection of shapes. We can divide the constructions into two groups. Some designers chose to build three-dimensional constructions (7 of them) and 3 of them decided to build a two-dimensional construction (lying on the carpet). Another criterion was the choice of the selected shapes. Most designers chose planar shapes (squares, triangles, polygons), while some chose spatial shapes.

There were also differences in the description of the construction. Some children first described the construction as a whole (e.g. “a house”) and then they focused
on the details. Others focused on the process of the construction creation. The children described how they put the shapes together (e.g. “first you put the square underneath a triangle, on the right next to it you put another square...”), etc. The description of the construction was very difficult both to explain and to understand the spatial relations. If the description was primarily based on these relations, it usually resulted in two different structures.

What can the children learn during the task? The task requires them to:

a) set the number of shapes that will be used in the construction or that will be necessary for the copy construction (the more shapes, the more difficult construction?), i.e. quantitative, pre-numeral ideas (quantity of a set)

b) be able to compare and contrast individual building shapes, to recognise the same shape, size and colour of the selected parts (triangle, square, rectangle,...etc., small, big,...etc.)

c) use manipulative skills in multisensoric perception, where eye sight is applied as well as touch.

Where did children fail? Regarding the difficulty of the task as well as the vast variations in task interpretation (two- or three-dimensional constructions), the children did not reach a successful result, i.e. identical constructions by both designers and builders. The builders sometimes did not recognise from the designers’ instructions if the constructions are two- or three-dimensional. Children who used three-dimensional shapes took longer time to finish their constructions. Thus the organisation of the whole group activities got more difficult. It was necessary first to let the designer choose the shapes, then his builder mate, and only then another pair could proceed with the shapes selection.

The success was essentially conditioned by mutual good coordination, communication, and the ability to get one’s ideas through between the designer and the builder. The problems in mutual communication became a serious obstacle in successfully finishing the task. Not all pairs were able to understand each other and to cooperate. Those who were thinking aloud and were commenting on their process thus created better conditions for mutual communication. Passivity in communication reduces the chances of understanding one’s own thinking as well as thinking of others, which evidently hindered succeeding in the task.

When realizing the activity some children probably got into conflict between deterministic decision-making on the best procedure (how to reach the goal) and egocentric adaptive decision-making on values (how to choose and set the shapes and colours in order to achieve the best visual impression).

Solving task helps to discover the relation between 3D and 2D reality models representing dual expression of the real world elements. Understanding the relations between attributes of individual 3D and 2D objects as well as the objects
themselves is considered a significant part of cultivation of the children’s geometry perception of the world (van Hiele, 1986). The shapes’ number and their form are given by the model. Anna created models of various difficulty, consisting of different number of shapes and different forms. The difficulty of some models could be alternated by the number of shapes, construction appearance as well as individual shapes forms. The activity organisation got easier as the shapes were available in sufficient numbers to all users and the designers could choose their shapes simultaneously, thus enabling the pairs to work simultaneously.

When realising the alternative activity a better task structure and its staging could have been prepared (which is a necessary condition of better adequacy and motivational task potential). The analysed task characteristic feature is joining intellectual operations with social and communication activities. Anna should have asked the children more questions and let them think more about the task, which could have been done better in both task variations.

At this stage the need for supervisor’s guidance turned out to be essential as well, e.g. by asking questions such as “Are there any other ways of doing it?” The aim was to encourage the development of student’s reflective skills in order to become independent in her self-reflection, leading to conscious learning from her own experience: “Watching the video helped me realize what all the essential things to keep in mind are when planning an activity. I enjoyed finding out how many different skills can be developed by a single activity and how much we can achieve with just one tool. When watching the video I also noticed children’s reactions and the possible ways of responding to them.”

Conclusion

For processing the materials for the creation of professional portfolio we used reflection of video recordings. We see portfolio in the same way as Beneš (2008), i.e. as a tool for professional development of teachers which later becomes an integral part of the lifelong process of professional training, which is an important condition of being a teacher. The portfolio expresses the attitude of the teacher to the profession, his/her competences, personal preferences and internal beliefs. The fact that the author collects the results of his/her work in a certain period, enables him/her to see his/her own progress in time. He/she comes back to what he/she lived through, managed or in what he/she failed. The process of working with portfolio consists of stages of material collection, selection, reflection and decision making of whether to include a particular piece of material in the portfolio or not.

The research outcomes clearly showed the great variety of attitudes of the twelve prospective kindergarten teachers to the construction and nature of the portfolio. The mathematical component of their portfolio contains a wide range of materials: written documents, games, video recordings, photographs, and other materials (Syslová et al., 2018). The ideas of activities included in the portfolios contain
various areas of mathematical pre-literacy: sorting objects based on given criteria, development of geometrical imagination during plays with didactic tools and building sets, links between mathematical and reading pre-literacy, application of regularities and rhythm, links between mathematical pre-literacy and artistic or musical activities, etc. In this way prospective teachers applied various aspects of knowledge acquired during their previous multi-area training. The form of their portfolio was influenced not only by their knowledge of elementary mathematics but also by their personal preferences of subjects and activities such as literature, technical skills, arts, music, etc. Inclusion of activities in our sample of respondents enabled us to “create links not only between pedagogical theory and practice but also between general pedagogical and didactic topics, topics of particular subjects and their didactics as well as other generally cultivating topics and issues” (Tomková, 2018, p. 215).

The ALACT model used in our research also allows for being confronted with a similar supervisor’s attendance model, the AAA (annotation–analysis–alteration). This model was used by Janík et al. (2013) on evaluation and developing of lesson quality by reflections.

The video recordings analysis allows many opportunities to focus on the didactic aspects in terms of pre-literacy:

“I realised that one activity develops more competences and skills, how many things can be developed by a single activity and how many things are related to mathematics.”

“Everything is related to everything, e.g. the children distinguishing things. Even this can be considered as an pre-literacy skill. They can distinguish objects, they can build from construction sets, cut something from a paper, count, …etc.)”

We think that in this way we can use didactic video case interpretation as a means of further development for teacher’s professional talents, such as diagnostic competence and professional perception.

References


POSING PROBLEMS AND DESIGNING TASKS TO PROMOTE TRANSFER OF LEARNING IN GEOMETRY BY TEACHER RESEARCHERS: THE CASE OF TESSELLATIONS

Tikva Ovadiya

Abstract

This study followed the process of seven teachers in the role of researchers (TRs) who were asked to design problems and tasks in tessellation for their students (Grades 1-4, 7) in a way that would promote transfer of learning from tessellations to geometrical or mathematical (e.g. multiplication) concepts, and vice versa. The study found seven features common to the problems the TRs posed to identify students’ transfer of learning and identified six levels of transfer expressed during the learning of tessellation. Teaching tessellations motivated both teachers and students to connect mathematical and geometric knowledge.

Keywords: posing problems, teacher researchers, transfer of learning, geometry, tessellations

Introduction

The main purpose of this study was to examine how mathematics teachers approach the challenge of posing problems and tasks to teach a new subject to their students. In general, teachers tend to rely on ready-made textbook problems, therefore, I asked them to teach tessellations, a subject not part of the curriculum of the grades under study (Grades 1-4, 7) and one they had never taught. Thus, they were forced to prepare problems and tasks on their own that were in accordance with their students’ present knowledge and would also complement the official curriculum.

The study’s secondary purpose was to determine how teachers, as “teacher researchers” (TRs) could research the concept of transfer of learning (ToL), first by constructing their own definition of the concept (they had learned the theoretical definitions of ToL) and then designing appropriate teaching and research tools.

A third issue was how the TRs observed their students’ understanding of tessellations in the context of elementary school geometry.

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Theoretical background

Posing problem and designing tasks

Designing mathematical tasks is an important research area in mathematics education. However, most studies take place in the context of professional development workshops that are aimed at enhancing problem-posing skills (e.g. Singer, Ellerton and Cai, 2015).

But problem posing plays a crucial role in the teaching and learning process, and Burkhardt and Swan (2013) pointed out the powerful role that multi-dimensional task-classification schemes play in the design of problems, tasks, and task sequences. They listed seven principles for “good” task design: adjusted to meet goals and objectives; contain essential mathematical ideas; are based on students’ previous knowledge; invite students to engage in them; encourage decision-making and creative thinking; include several representations; and create opportunities for discussion and collaboration.

Barabé and Proulx (2015) wrote a review summarizing (explicit and implicit) studies in the field. They recommended constructing additional theoretical frameworks to refine the research, however, to date, I have found no study that follows teacher-designed tasks based on their own research of ToL.

Transfer of learning

The literature reveals little agreement about the nature of ToL or the extent in which it occurs. However, for the purpose of this study, I followed Lobato’s definition: a person’s ability to generalize learning so that a subject studied in one mode can be applied to another (2003). Her studies were unique in that they examined the process (i.e. the stages) of acquiring knowledge, skills, and performance, as opposed to other studies that 1) merely focused on the final performance (e.g. test results), on theoretical problems, and on subject matter instead of the individual; 2) were conducted in laboratory settings (not “in real life); and 3) did not allow for the different forms of ToL that were a result of an individual’s imagination and socio-cultural contexts (Lobato, 2006). According to Lobato, ToL is a dynamic process, and the nature of thinking and the knowledge transferred is more important than final performance. I.e., “production” is more important than “implementation” (Lobato, 2014). In her 2012 paper, she gave precise guidelines for research using a student-oriented approach and how to design assignments to understand how students connect what they learn.

Lobato, Rhodehamel, and Hohensee (2012) offer an alternative definition of ToL and suggest ToL may be determined by observing individual cognition along with the structure resources of the student’s social activity. They also suggest considering the extent of the student’s attention, including what is paid to “competing” resources during learning, such as the character and mathematical properties of the task, the learning situation, other students in the classroom and
their contribution to the student’s individual learning, etc., and then observing and identifying the student’s reflective abstraction and generalization of knowledge.

This study focuses on analysing the findings and interpretations of teachers who were asked to research ToL in accordance with Lobato’s definition.

**Understanding tessellations**

The earliest educational study using tessellations was done by van Hiele, who developed a model of geometric reasoning and, following an entire school term devoted to leading students in the construction and analysis of tessellations, reported a significant increase in her students’ ability to reason geometrically (van Hiele, 1959). Current studies (e.g. Eberle, 2014) also point to aesthetic criteria that motivate students’ explorations in the creation of tessellations and influence their evaluations. Moraová, Novotná and Favilli (2018) demonstrated that exposure to tessellation activities promotes mathematical knowledge and creativity for both students and teachers alike. Since tessellation involves the perception of shapes and relationships, educators must teach the use of scaffolds to enable progress from the initial stage of mathematical concepts to independently demonstrating conceptual understanding.

To date, there appears to be no study that monitors the perceptions and performance of teachers who both design tasks to teach this topic and conduct a study of ToL from tessellations to mathematics and geometry and vice versa. Thus, this study comes to answer two questions: 1) What characterizes the tasks, task sequences, and problems that TRs pose with respect to tessellations to promote students’ ToL? 2) How do TRs understand their students’ ToL process?

**Method**

For this one-year study, I followed seven teachers studying for their MA in teaching (specializing in mathematics education) and concurrently teaching in schools. Six taught grades 1, 2, 3, or 4 (25-30 students per class) and one, grade 7 (18 students). The teachers were asked to study tessellations independently and then use their knowledge to prepare lessons, design tasks, and pose problems for their students in such a way that 1) the information the students were taught about tessellations would be transferred to other concepts, and 2) they could study precisely how ToL manifested itself from tessellations to the other mathematical aspects and vice versa.

As mentioned, none of the teacher-researchers (TRs) had taught this subject previously, nor was it part of the curriculum (for these grades), thus TRs were forced to design their own teaching material. Before commencing, they were taught three aspects of teaching and research management: 1) constructing age- and grade-appropriate lesson plans; 2) using worked examples; and 3) writing hypothetical instruction scenarios (based on “Learning through Acting” principles, Simon, Kara, Placa and Avitzur, 2018) before each lesson. After each
lesson, they were to note in their journals their students’ reception of the lesson, if/how ToL occurred, and any other observations.

**Data sources and analysis**

Data was harvested from 1) the TRs’ journals, and 2) video and audio documentation of their teaching and the mentoring processes. First, the problems and tasks they designed were analysed for complexity, unfamiliarity (non-routine), technical demand, and student autonomy (based on Burkhardt and Swan, 2013). Then, they were classified as expert (complex tasks requiring advanced problem-solving strategies); apprentice (involving several aspects of mathematics yet structured so that all students can deal with the problem); or novice (tasks focusing on one specific concept or skill). I also identified unique characteristics of the problems. Task features were identified and any common potential or actual generalizations with respect to transfer levels noted.

Data was examined to elucidate the TRs’ teaching goals, and common considerations and catalysts were gathered. Also, I studied their perceptions of ToL and how their instruction implemented it (based on Lobato, 2012; 2014), noting which generalizations they intended to develop and which generalizations they identified.

**Findings**

**Features of problems, tasks and task sequences:** Seven features were identified: (a) presenting examples and non-examples; (b) demonstrating a variety of solutions (a solution applicable to every knowledge level); (c) enabling autonomy by providing guidance and scaffolding for independent work; (d) building critical properties gradually; (e) investigating specific features of a tile (e.g., “symmetry”) once the overall concept has been assimilated; (f) constructing tiles (which requires higher skills than recognizing one: students must manipulate and involve additional concepts, allowing the TR to evaluate concept perception); (g) connecting the characteristics of various shapes and relationships (for a deeper understanding of mathematical relationships). Series of tasks that included all or most of these seven features could indicate generalizations that indicated whether ToL occurred.

**Transfer levels:** Analysis of the TRs’ journals for their understanding of ToL of tessellation to other concepts indicated that, in general, the TRs were aware of six ToL levels, as follows: 1) “visual generalization” (e.g., the ability to draw an image of tessellation without understanding all the geometric aspects); 2) “local generalization” (understanding at least one condition, e.g., that all tiles are identical); 3) “generalization of a critical feature” (observing the unique features in a tile that make it suitable, and transferring this observation to other tiles); 4) “analysis of geometric shapes and generalization of common features” (analysis and classification of triangles and squares, and generalization of the principle that if a square contains two, three, or four tiling triangles, it is a tiling
square); 5) “ability to construct tiling” (forming a suitable shape not merely as a technical skill but as the result of generalization: e.g., using a tile previously found suitable and then creating a variety of tessellations using symmetry, sliding, rotation, mirroring, etc.); 6) “developing relationships between different mathematical concepts” (solving tessellation problems using multiplication and vice versa). This indicates the “ultimate” generalisation of the complex concept of tessellation and the geometrical aspects associated with it.

Specific examples

Example 1 (Grade 2): Which expression can describe this tessellation? Explain. $7 \times 5; 6 \times 7; 6 \times 5$.

**Task level:** Apprentice, but novices can solve it as the “6” is obvious in the basic figure. The illustration promotes autonomy. **Task features:** (b): locate repeating tile without remembering all tessellation features; (c): identify that each tile is composed of six irregular pentagons; (d): note that a 6-polygon tile appears 7 times; (e) & (g): recognize the shape and identifying that it can be described using geometric language (tessellation) or multiplication or both. **Transfer levels:** Level 3 (critical feature of the multiple): One tile repeats 35 times, or a tile composed of 6 pentagons repeats 7 times, or the 6-pentagon “flower” can be broken down into another structure and found seven times. Also, level 6 (obvious ToL from multiplication to tiling and vice versa): “This can be described and analysed by multiplication, and multiplication can be explained by it”; “The tile can have several variations, as can multiplication.”

Example 2 (Grade 2): Cover the black hexagon with the different polygons. Determine the area of each polygon. Use the tiles to form a trapezoid, a rhombus, and a triangle. (All triangles are the same size).

**Task level:** Expert. **Task features:** (b) (c): understanding tile properties; (d), (e), (g): connecting tile properties with tessellation; (g): connecting tile properties with area. **Transfer levels:** Several levels of transfer were expressed. E.g., level 2 (“the tile is a triangle, with it I can compute the area”) progressing to level 3 and then to 6 (“Critical feature of the black area is that it is composed of triangles, the unit tile is a triangle, and each given shape is a complex of triangles. I need to calculate the area to be tiled. If it is divisible by 1, I can use triangular units, if by 2, rhombus tiles, if by 3, trapezoid tiles.” This answer demonstrates exclusion, inclusion, and generalizations.

Example 3 (Grade 4): Below are three types of tiling: “non-tiling” (circle), regular (trapezoid), and advanced (hexagonal). Use any one to construct a tessellation.
**Task level:** Novice, but also appropriate for experts as choosing a complex tile will increase difficulty.

**Task features:** (a): present examples and non-examples; (b) and (c): choose a tile as simple as a square; (c) and (d): autonomy in choice and actions but guided by examples; (e): investigate the tessellation by features such as reflection symmetry as in the trapezoid; (g): gradually build knowledge about the relationship between properties of the tile and the characteristics of tiling. Optional (f), if the student constructs an original tile.

**Transfer levels:** Generally, novices chose an equilateral triangle or square (levels 1, 2): “This shape is a tile that can continue in an endless tessellation.” Experts chose a hexagon or pentagon. However, those who chose the hexagon were able to do the tiling but explained themselves like novices. Some chose the pentagon, not realizing it was not a tile, but after further investigation, understood that its properties did not conform to a tile’s, but then used them to build a hexagon and were able to explain why this was appropriate for tiling (levels 3, 4). Some generalized the features of an individual tile and then generalized the features of the pentagon as non-tiling. Others determined the relationship between geometric properties of shapes and critical properties of tiling. Others constructed a tile that was a right-angled trapezoid and found it suitable, some used two trapezoidal rectangles, justifying the rectangle as a tile (level 6).

**Discussion and conclusions**

In this study, I designed an environment where TRs could not rely on mathematics textbooks and thus had to upgrade their geometric knowledge, understand the development sequence in understanding the concepts of tiling (tessellations) and ToL, and come up with (original) ideas for implementing teaching tools and research tasks.

Clements and Sarama (2014) measured mathematical proficiency according to five features (based on Kilpatrick, Swafford and Findell, 2001): conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Comparing these five features with the seven task features (a-g) observed in this study demonstrates that the TRs did relate to all five, albeit without formal knowledge of them. Indeed, the TRs themselves solved problems they found on the subject and, based on their understanding of their students’ present knowledge, prepared lessons that included series of problems to develop their students’ conceptual and procedural understanding. At the same time, they designed a research study to examine logical generalizations of concepts, reiterating the suggestion by Abramovich (2015) that mathematical problem posing can serve as a link between algorithmic thinking and conceptual knowledge.
Clements and Sarama (2014) claim that learning trajectories have three parts: a mathematical goal, a developmental path to reach that goal, and a set of instructional activities appropriate to the students’ level of thinking to effect the path. The TRs employed all three while noting different generalization levels for each problem. This illustrates that the hypothetical scenarios did serve as catalysts for posing relevant problems according to different stages of learning, generalization, and ToL.

Similar to Hiebert (2013), who described conceptual understanding as a network where the relationships represented by the links are as important as the items themselves, I found that ToL occurred at all levels, since they all included items that had to be analysed and relationships that had to be understood and analysed. At the highest level (level 6), students had to create mathematical connections between items that do not necessarily invite linking, but the link develops the conceptual understanding of each one: for example, the relationships between multiplication and tiling, between area and tiling, and between right-angled trapezoids and rectangles. Clements and Sarama (2014) argue that children can work at more than one level at a time, since levels are not “absolute stages” but are “benchmarks” in complex system of growth and represent distinct ways of thinking. So, another way to think of “levels” is as a sequence of different thinking and reasoning patterns. Children are continuously moving between levels. In the TRs studies, a particular mathematical problem could be solved differently by students at different levels of thinking. In consolidating the six stages towards transfer, it was found that students often moved between stages on the same problem and at the same learning stages. The ToL ranking helps teachers (as researchers) understand (notice) the developmental process, as Lobato (2012) defined “noticing” to be an alternative transfer of learning process, even if unable to unequivocally determine the development sequence.

In this current study, participants presented a subject that, in the formal national curriculum, was appropriate for higher grades, and therefore the TRs had to adapt problems and instruction principles (e.g., the “trajectories approach”) to their students’ levels. However, their research attested that the development of perceptions of forms, attributes, and relationships can occur at a younger age than usual: a result of having included each of Kilpatrick and colleagues’ (2001) five elements of skills and presenting them in an environment that promoted different transfer-of-learning levels.

This study has two theoretical contributions: defining seven characteristics of problems for promoting ToL and defining six levels of ToL. In addition, it offers three practical contributions: 1) features of problems and tasks to promote ToL; 2) levels of ToL that students undergo while solving such problems; and 3) a demonstration of how lessons in geometry are designed through a series of steps that present suitable problems, study the students’ process, and then builds the next steps based on the observations.
References


PROMOTING THREE-DIMENSIONAL SPATIAL PERCEPTIONS OF PRISMS: THE CASE OF ELEMENTARY-SCHOOL STUDENTS USING AR TECHNOLOGY

Tikva Ovadiya, Osnat Fellus and Yaniv Biton

Abstract

In this paper we report on a study of the effects of propaedeutic approach related to development of understanding of multiple representations of fractions. Grounded in the literature review on the children’s understanding of representations on one hand and problem posing on the other, we analyze students’ achievements in solving problems, particularly examining transfer of knowledge in solving non-realistic problems and problems in real context involving different representations of fractions. A sample of fourth grade school children were enrolled in the experimental program in regular school setting. The results obtained in the experimental study show that the integration of learning content in propaedeutic learning has significant impact on the performance of students in non-realistic problems as well as in problems presented in a realistic context. Our discussion of results point for the need for further examination of the effects of using multiple representations of math concepts on children’s adeptness in solving particular types of problems.

Keywords: fractions, representations, non-realistic context, real context

Theoretical background

The relationship between understanding geometry and learning mathematics has recently been highlighted by a growing number of scholars (Sinclair and Bruce, 2015; Uttal et al., 2013). This relationship has been suggested as far back as four decades ago by Tahta (1980) who explains how geometry is integral to mathematical understanding and how treating them as separate entities may cause substantial problems in the learning process of mathematical ideas. In this context, scholars have shown that many students see the learning of geometry as a difficult subject (Clements, 1998; Mulligan, 2011, 2015) but that improved visuospatial skills and reasoning are strongly associated with a better understanding of concepts and knowledge in geometry (Casey et al., 2008), and that the use of digital technology contributes to improved spatial awareness on the one hand and leaning geometry concepts on the other hand (Battista, 2001). This strong association between the use of digital technology and the learning of geometry generated advanced tools that allow learners to enhance their visuospatial skills by harnessing movement and gestures to the process of learning of geometrical shapes. In this paper, we focus on the affordances of the use of AR in learning about prisms.

In a recent publication on the use of AR, Young and Santoso (2018) have summarized the contexts, uses, limitations, and main findings of research on AR

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in the last 15 years. Drawing on this work, it is yet unclear how the technology of AR specifically contributes to the learning of concepts in geometry or to the understanding the process of solving a problem in spatial geometry.

Spatial geometry includes five features: Spatial awareness, spatial visualization, rotations, animation, spatial relationships, and spatial orientation (Clements, 1998). The main object of teaching geometry is to develop these aspects. Researchers suggest that learning geometry using AR enhances spatial thinking and skills. For example, Kaufman and Schmalstieg (2003) developed a learning environment for teaching spatial geometry that included AR technology. Their purpose was to demonstrate geometrical shapes and their spatial properties in order to make it possible for the students to learn about the shapes not only visually but also through their shape-specific properties.

Kaufman (2006) analyzed data collected from more than 300 students who were using a dynamic geometry environment that included AR as they were learning different topics in geometry. Kaufman showed that the very opportunity to engage in spatial rotation of geometrical objects not only provides teachers with a better understanding of students’ difficulties in spatial reasoning but also advances learners’ spatial awareness of tangible objects. Radu et al. (2015) developed an AR-based game for young children that provides a structured framework for the development of visuospatial skills. Carefully attending to users’ comments and reactions, the researchers used formative assessment to further develop the platform and the tasks to ensure a better reflection of the different levels in knowledge and of the curriculum requirements.

The purpose of the study

The purpose of the study is to characterize the processes Grade-6 students employ when solving a problem on prisms using AR technology and to compare these with problem-solving processes among students who do not use the AR technology to answer the same geometrical questions.

Research question

Against the backdrop of the affordances that technology offers in teaching and learning geometry, we are interested in identifying patterns in solving geometrical problems among Grade 6 students. Specifically, we asked the following questions:

1) Whether, and if so how, the integration of the AR technology advances Grade-6 learners’ spatial skills and reasoning as they answer spatial-related geometry problems?

2) What typifies problem-solving processes through AR technology in comparison to problem-solving processes without AR?
Methodology

In order to identify differences between groups, we employed the following three stages:

First stage

The purpose of the first stage was to comprise an experimental group that would use the AR technology as the students learn about prisms and their properties. In order to create such a group, we wanted to create a baseline through which we can capture a snapshot of students’ knowledge of the subject of prisms and their properties. Twenty-two boys and 22 girls studying in Grade 6 in a school that is located in the North of Israel took a test on the subject of prisms. Students were given summative tests at the end of the process.

The summative test included four tasks on the topic of prisms in which the first task asked students to identify the area and volume of a cuboid following a change in its orientation and a removal of a cube from different locations in the cuboid. The students were asked to formulate the effect of these actions on the volume of the cuboid. The second task provided the students with different geometry nets and the students were asked to determine whether or not the nets were of prisms. In the third task, students were presented with different solids and were asked to identify which were prisms and what the shape of the base was. Finally, the students were given cut prisms and were asked to identify whether the newly cut shapes were prisms. Table 1 presents the results from the test:

<table>
<thead>
<tr>
<th>Task #</th>
<th>Section</th>
<th>Girls</th>
<th>Boys</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Correct answer</td>
<td>Incorrect answer</td>
<td>No answer</td>
<td>Correct answer</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>15</td>
<td>5</td>
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<td>1</td>
<td>2</td>
<td>2</td>
<td>15</td>
<td>5</td>
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<td>1.1</td>
<td>11</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>2.1</td>
<td>1</td>
<td>16</td>
<td>5</td>
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<tr>
<td>1</td>
<td>1.2</td>
<td>4</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>2.2</td>
<td>0</td>
<td>16</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>1.3</td>
<td>7</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>2.3</td>
<td>0</td>
<td>16</td>
<td>6</td>
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<tr>
<td>1</td>
<td>1.4</td>
<td>3</td>
<td>13</td>
<td>6</td>
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<tr>
<td>1</td>
<td>2.4</td>
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<td>12</td>
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<td>2</td>
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<td>11</td>
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<td>10</td>
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<td>4</td>
<td>18</td>
<td>1</td>
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<tr>
<td>2</td>
<td>5</td>
<td>12</td>
<td>7</td>
<td>3</td>
</tr>
</tbody>
</table>
As the table shows, boys did better in most of the tasks. In addition, in order to organize the data, each student’s work was treated as a separate, contained data section. We next decided on cut-off points to be able to create the baseline of each student’s achievement. The following four quarters were created to represent different levels of achievement: Quarter I included data sections of up to nine errors; Quarter II included data sections of between 10 to 18 errors; Quarter III included data sections of between 19 to 27 errors; and Quarter IV included cases of 28 errors and more.

Second stage
Following the categorization of the data sections to create the baseline, one-to-one interviews were conducted with three students from the Quarter IV, two from Quarter III, four from Quarter II, and four from Quarter I. Altogether, there were 13 half-hour interviews of which 7 were with boys and six with girls. The rationale behind selecting students from each quarter was to see how the use of AR changes thinking processes in students from different baseline groups.

Third stage
Given that the unit is in its developmental phase, it was important for us to conduct interviews to better understand the students’ thinking processes as they were solving problems with AR and without AR. The third stage of the research
included the 13 interviews that were later transcribed verbatim. Each interview included two parts. In the first part, students were asked to carry out a few tasks working on prisms without using AR. In the second part, students were asked to solve problems on prisms with AR. The following tasks were given to students:

In each of the following cases (Figure 1), a prism was cut by a plane to create two shapes. Determine whether the two new shapes are prisms. Circle the cases that are prisms. For each new prism determine the shape of the base.

![Figure 1: Cutting prisms](image1)

An additional task (see Figure 2) was to identify prisms out of an array of four shapes. We wanted to see what would be the reasoning processes that students would use to determine which of the given shapes is a prism and how would the students justify and evaluate their answers as correct. Here too, the students were asked to justify the selection of the shape and determine the shape of the base.

In both sets of tasks, students would look at the shapes on the paper, and would use the AR technology only when they could not provide an answer.

![Figure 2: Selecting prisms](image2)
In a third task, the students were given a variety of polyhedrons and were asked to identify which were prisms and what was the shape of their base. In Figure 3, you may notice that not only are the shapes lying on one of their faces rather than on their base but also that their respective base is not a shape students are familiar with such as rectangular, a square, a triangle, which increased the level of difficulty of the task.

![Figure 3: Identifying prisms](image)

As indicated earlier, during the second part of the interviews, the students used the AR application. The students were asked to justify their responses as they were reasoning and thinking aloud.

**Results**

We would like to remind the reader that the purpose of our study was not to focus on students’ errors or on concepts of shapes but on students’ sense making processes and on addressing misconceptions as students were using AR. Following data analysis we found that when students were working without the AR, they engaged with two types of reasoning: first identifying the bases of the shapes, when they could not find the answer; second, they would look at the surface area and try to identify the base from there. These two types of attempts to find the answers were carried out as the students were making statements that reflected attempts of unfolding the shapes in their imaginations.

We categorized these *judging by transformation*. However, when the students could not provide an answer, they would “unfold” the shape and rotate it using the application to identify the base. We realized how the acts of transformation and rotation are significant for understanding geometrical shapes and developing
spatial awareness. In addition, using AR, we saw that students preferred keeping and analyzing the shapes as whole (3D) and only when they could not find the answer, they would try and understand it through creating geometrical nets (2D). In other words, the students’ motivation was to remain in the 3D level of thinking and try to understand the shape before “retreating” to the 2D space of the geometrical net.

![Figure 4: Sliding, shifting, and moving from 3D to 2D using AR](image)

Looking at Figure 4 from right to left (as reading Hebrew dictates) we see an example of a student’s work with AR rotating and moving a shape before “unfolding” it. In regard to using the AR, we noticed that all students regardless to their baseline results used AR. The ones whose baseline results were in Quarter IV tended to use AR and then apply transformations to the shape whereas those whose baseline results were in Quarter I tended to not use AR as often.

**Discussion**

Work on AR (Martín-Gutierrez, Trujillo and Acosta-Gonzalez, 2013; Young and Santoso, 2018) tends to focus more on the development of the visuospatial skill and less on the manipulation of shapes in 3D. Based on the dialogue we had with the students as they were doing the tasks, we tentatively suggest that students prefer to understand shapes through their spatial properties, which may explain the first choice of action to move shapes around. Secondly, students use a sequence of actions and rationalize their actions as they are engaged in sense making of the visuospatial context. Thirdly, as far as the students are concerned, rotations and moving shapes in space precedes the “unfolding” of shapes to create nets. In regard to the differences between boys, while these require further investigation as to the nature of the differences, the results support work done by Yilmaz (2017) and Merrill, Yang, Roskos and Steele (2016).

The tasks that were given to the students where worded in such a way that require thinking and deciding which action to carry out with the AR. The tool, then, helps in choosing how something will be viewed and observed. Drawing on Koichu (2018), we argue that the solution of a problem is the meeting point of an array of possible ways of solutions. The contribution of the current study is in showing that one of these available tools is the opportunity and the possibility to choose in AR.
One of the conclusions we can tentatively draw is that in the first phase of the research, students need to receive simple problems to make it possible to identify simple actions to carry out using AR. Indeed, we realized the first actions the students carried out were movement of the shapes – an action that can be used with physical objects and is not restricted to the virtual tool. However, the possibility of “unfolding” the shape into its net is tool specific as it allows students to examine shapes from different perspectives. Results illustrated at what stage students retreated from the 3D to the 2D space after they could not find the answer by simply rotating the prism.

The next step that is planned for our research is to create opportunities to solve problems by using AR rather than using 2D shapes and looking at tasks that go beyond mere definitions and recognition of prisms to include relationship between properties between and within shapes.

Acknowledgement: We would like to thank the CET (Center for Educational Technology), Israel for the current development of the AR-accompanied Grade-6 textbook, Shvilim Plus (2019), and for providing permission to conducting this research. The AR app was imagined and created by reCET unit in CET.

References


OPPORTUNITIES OF USING RADIO RESOURCES IN MATHEMATICS EDUCATION AS A SPECIALIZED LANGUAGE SUPPORT

Franziska Peters

Abstract

This paper is focusing on auditory learning and the use of radio features for specialised language support in mathematics education at the primary school level. Radio features with its relevance in everyday life can serve as a natural auditory learning material for children. Thus, we developed various possible applications of radio resources as auditory learning material and implemented them in various primary school classrooms. One example is described and analyzed in this paper. The aim of this research is not only to investigate the effects of auditory material on the learning procedure in general, but also to investigate how auditory material can serve as a provided language model. First results show that auditory material can indeed be effectively implemented in teaching practice and reveal just how these effects look like.

Keywords: auditory learning, radio resources, design science, language support

In 2015, the Department of Mathematics Education at the University of Giessen started a first project in cooperation with a regional radio station “hr2 – Hessen Radio for Culture”. This radio station developed, inter alia, a series of radio broadcasts on mathematical topics for the primary school level, collected in the multimedia offering “Kinderfunkkolleg Mathematik” (www.kinderfunkkolleg-mathematik.de). Within this collaboration project, future mathematics teachers developed auditory material for use in mathematics education at the primary level (for more information about this see Peters, 2018).

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In a second project, future mathematics teachers planned teaching units for performance in schools, using auditory educational material as a central element. It was important to interrupt longer listening phases and repeat the segments while giving listening tasks in between. The auditory material could be used as preparation of a topic or as a base for discussions, as well as for explaining and deepening new content or repeating previously covered topics. The units were realised in different schools, reflected on in the seminar, optimized and then turned in. After being corrected and edited, the units were then allocated for the download centre of the “Kinderfunkkolleg Mathematik”, as accompanying material.

Based on this project, I started developing research aims and the design of my pilot study to further research on the effects of radio resources as auditory learning material. The material, used in this study is developed and owned by professional radio authors and producers who received help from experts in our university department.

**Auditory learning**

In order to research the effects and the use of auditory material, one has to understand how auditory learning is working. According to Baddeley (2007), in the working memory visual and auditory information are processed in different sensory channels. Auditory learning reduces sensory impressions on the visual channel, so that the auditory channel is required and therefore trained more. Acoustic information is processed through various memory processes. The echo memory saves mostly unprocessed sensory impressions for a short period of time. In a next step, the working memory with its limited capacity processes the acoustic information and saves it in the phonological store. To reproduce written information in oral form, one has to have recourse to the phonological form of that information. Thus, written information has to become encoded into phonological form. This takes place in the phonological loop. Spoken information on the contrary can be saved phonologically in the phonological store without intermediate decoding. The functioning of the phonological loop is affecting language competence and performance. Thereby it is affecting mathematical competence as well.

The working memory can also build temporal connections that are necessary for remembering the beginning of a sentence while hearing the end of it (Leuders, 2011). According to Leuders, an increase in efficiency of the working memory is a training effect. Since the processing of auditory information takes place in the working memory, one can also assume that the increase in auditory learning efficiency is a training effect. This is a fundamental trigger for my interest in research on auditory learning materials in mathematics education.

Another important aspect is the Cognitive Load Theory by Sweller (1994), which says that the capacities of the working memory are limited and should not be
exhausted by extrinsic factors. One of those extrinsic factors is reading. Reading difficulties can exhaust the working memory and lead to not understanding mathematical content as well as not being able to solve mathematical tasks. Keeping mathematical concerns in the center of learning processes while reducing extrinsic factors, is one important approach for auditory learning.

Auditory learning material

As a first function, it can be stated that auditory support can aid children with reading difficulties to understand mathematical contents and tasks without the need of having to read coherently and extract the meaning. This, however, does not mean that reading should be replaced by hearing. Auditory support should only be provided if needed. According to the presentation of the content, auditory material can counteract transitory learning through possible repetition. Teachers and students are able to decide individually which sequences they need to listen to again (Peters, in print). Those sequences can hereby serve as verbal language examples in the sense of Scaffolding (Gibbons, 2002).

Another function of the use of auditory material is the development of active listening skills. Such competence is a primary requirement for education, but it is seldom supported or even trained (Pimm, 1987). In Germany, education standards for the subject of German language make it very clear: These require, not only the competences of reading and writing, but also speaking and listening. Speaking is required to be consciously organized, while terminology is to be trained and (the use of) language is to be examined. Regarding listening, children are to listen attentively and perceptively, while registering others’ statements and constructively dealing with them. Of course, listening also takes place in the form of frontal teaching, but this kind of listening is difficult for most students. This is why supportive elements for auditory learning are necessary – both in frontal teaching and by means of group work.

Mathematics register acquisition

The mathematics register can be seen as “consisting of technical terms, diagrams, and grammatical constructions, such as logical connectives” (Meaney, Trinick and Fairhall, 2012). These elements are learned throughout the entire school process and students are constantly “introduced to further layers of meaning for the terms and expressions they already know” (ibid.). Leung (2005) points out that learning vocabulary in mathematics means learning both formal and semantic features of words in various contexts, and involves thinking with and through the concepts. Furthermore, she calls it an “incremental activity” (ibid.), stating that meanings can develop and expand.

The model for mathematics register acquisition (MRA) was developed to categorize teachers’ strategies in teaching the mathematical register (Meaney et al., 2012). It is based on Gass’ model of second-language acquisition (1997). The MRA model is divided into four stages (Meaney et al., 2012). The first stage is
“Noticing”. In this stage, teachers introduce new terms or expressions, use them frequently and then encourage students to using them as well. The second stage, “Intake”, describes the process of understanding. Students start to explore and work with the new terms. In the next stage titled “Integration”, testing, feedback and modification takes place. Students have a good understanding of the new term and are responsible for using it, but might be supported or reminded of their knowledge. In the last stage, “Output”, there is a fluent use of the new terms. Teachers do not need to support, but should provide activities where the use of these terms would arise naturally.

The control of teachers and support through scaffolding is only necessary in the first two stages; in the final two stages, students gain control. The teacher only needs to provide opportunities to use the new terms. Regarding auditory material, we can see that it can be of use in the first two stages. It can be used to introduce new terms and to repeat them frequently in the stage “Notice”. In the stage “Intake”, it can serve as a language model with which students can work.

**Specialized language support through auditory material**

Auditory material can be called language driven media, a term that Ritterfeld and Langenhorst (2011) use for audio plays. The reception is based on language procession and there is no alternative visual input. Not-acoustic images cannot compete with the auditory input. Thus, it is possible to exclude the visual concept and focus on the auditory input. Besides that, students have to build up the visual concepts themselves without being supplied with a correct representation to fast.

As already mentioned, the learning of a language can be supported by training listening competence. Active processing is important for meaningful processing and for memorization of what has been heard. Reasonable and profitable use of auditory material is needed for these processes, for example good embedment, listening tasks or segmenting principle (Rink, 2017). As acoustic representations are volatilely, there is the need of adding opportunities to document the content of the heard and results of the related tasks within the teaching units. By these means, specialized language support can be ensured. Following these ideas, my research can be focused on the evaluation of auditory educational material in various settings – particularly regarding possible learning effects. The main interest of this research is the use of radio in mathematical education for specialized language support at the primary level.

Prediger and Krägeloh (2016) are referring to a model of three registers relevant for mathematical learning (everyday register, school register and technical register). This model illustrates different levels of verbal representation and how they are connected or built onto each other. Particularly interesting for my research is the question how children can be led from everyday register to school or even technical register.
School register is an important and necessary factor for successful learning in mathematics. It is a shared language basis and helps with explaining, describing and justifying (Götze, 2015). However, children do not bring this type of language to school with them. It must be learned, like registering a new language. This applies not only to children with special needs in language development but to every other child. That is the reason why they need linguistic models to develop educational language and to fill terms with representations. These linguistic models are scaffolding onto which children can lean (Gibbons, 2002). Lexical storages, which only include words, are not sufficient, as new terms must be used in whole phrases and sentences. According to Götze (2015), language acquisition is, in practice, merely a continuous learning process. There can be setbacks and sometimes children express themselves better in written than in spoken language. This is because everyday register is predominant in spoken language and, oftentimes, deictic expressions are used. This is valid for children as well as for teachers – even if they do so unaware and unintentionally. At this point, auditory educational material could be a useful and profitable addition.

If radio features are to be used as such educational material, it has to be analyzed, evaluated and then applied in school context. For the first step, the analysis of the feature, various criteria of a ‘good radio feature’ have to be considered. Radio authors emphasize the importance of writing for listening which leads to inalienable principles. Klug (n.d.) underlines a clear structure, simple sentences, linear provision of information, summaries, repeating important terms etc. through redundancies and of course a varied composition including different voices, sounds etc. This is only an extract of many criteria. For my work with radio features in mathematics education, I obviously have to add criteria as mathematical register and correctness, good examples and so on. After analyzing the features in this way, I can use them in classrooms to research on my actual interest as described in the following.

**Research questions**

The aim of my research is to find out in what way auditory learning material could be of use for language support in mathematics education? For this cause, I want to ask how auditory material, as a language model, can stimulate the development of the school register and how auditory material can support listening competence. In a second step, I want to research what a profitable use of such material could look like.

**Methodological approach**

Regarding the data collection, I have decided to us design research based on Wittmann’s Design Science (1995). Subject of the Design Science is the construction and research of teaching concepts, including accompanying theories. According to Wittman, this science is a practice-oriented core area for mathematical education, since it refers to the construction of artificial objects
(teaching concepts, curricula etc.) and the research on possible effects in different educational settings.

**Didactical design research**

Based on the Design Science, Prediger developed the model of design research (Prediger et al., 2012). The aim of this method is to effectively implement innovations for educational development in the teaching practice and empirical research, carried out under realistic conditions. In order to do this, one has to undergo a cycle of research multiple times. The cycle is composed of different phases which focus on either design or research questions – complementing each other. This way, the design process is leading to design results while the research process is leading to research results.

The cycle starts with the specification and structuring of the learning subject. In my case, this would be the used auditory material, which will be specified later in the explanations of the pilot tests. Based on this, a design is to be developed for the specific learning topic. In my case, a teaching concept and a teaching unit for a fourth-grade class was designed using auditory material to elaborate mathematical content in class. In a third step, the developed design was performed by means of a design-experiment. In this phase, the concept was tested in teaching practice, data was collected by filming and said data was evaluated. Based on the analysis of this data, local theories about the learning subject and the teaching concept can be developed in the last phase. The local theories are the starting point for next rounds of the cycle, in which they can help to optimize the learning subject and concept. In this way, after a few cycles, we will not have a perfect teaching concept or representative research results, but rather new and tested theories on the use of auditory media for specialized language support. By now, I finished the first cycle and am currently using my initial experiences to develop local theories, as well as structure the learning goals and teaching concepts anew for the second cycle.

**Pilot testing**

For the pilot testing, I designed a teaching unit in a fourth-grade classroom with four 45-minute classes. A teacher, who was specially trained for this project, instructed the unit. The topic was “Probability and Random Experiments”, based on the radio feature: “Wann ist ein Spiel fair?” (When is a game fair?) from the Kinderfunkkolleg Mathematik (https://www.kinderfunkkolleg-mathematik.de/themen/wann-ist-ein-spiel-fair). In this feature, four students are planning to play a game in order to make a decision. They realize the need to test the fairness of the game, start an experiment to do so and find out that the game is unfair, since they don’t have equal chances. While testing the game and talking about the mathematical problem, they use school register and mathematical terms. Thus, the students in the class that was observed and filmed were confronted with new terms in a playful way. The overall aim of the unit was the conceptual
development of “fairness” (through language) while the linguistic aim was the understanding of the following terms which were presented in the radio feature: double, street, probable, option, coincidence, fair, unfair, unsafe, unlikely, likely and safe. Those aims work together: language development can help to develop the concept. Throughout the entire day, both the class situation as well as the working phases in smaller groups were filmed. This way, enough data was gathered referring to the individual processes of the children’s speech development, while at the same time it was possible to research on what a profitable use of this material could look like. For the second testing, these experiences can be used to improve the unit and the use of auditory material.

The first lesson began by listening to the first part of the radio feature in which two children were arguing about the fairness of a dice game. In class conversation, the students repeated the content of the heard, reviewed the game and tested the fairness of that game in various steps that built on one another. In between those steps, more parts of the radio feature were presented, and a lexical memory was collectively developed based on the content of these features. In the second part of the day, the students verified the chances of winning and determined the fairness of various other games during learning stations. Hereby, they had to transfer their acquired knowledge whilst using the structures of language and reasoning they had been offered and trained through the radio feature. The stations were about wheels of fortune, about card games, about dicing with six- or ten-sided dice, about throwing different amounts of reversible tiles, about picking different colored balls from an urn and so on.

In groups of two, students chose a game for which they would be experts. Each group had to test every game, but they only had to work on a worksheet for their expert station. On that worksheet students had to describe and to reason whether or not that specific game was fair or unfair. If the game was unfair (which all of them were), they also had to explain why the game is unfair and how one could make this game fair i.e. how to increase the chances of winning for the losing card, colour etc. so that the chances become equal. After the learning stations, students ended the unit with the presentation of their “expert stations”. Each group presented their station, outlined how they examined the game and shared their results. They also presented a solution on how to make the game fair.

**Initial experiences**

Initial observations showed that students were highly concentrated while listening to the radio features. Due to the fact that students know audio-plays from their everyday life and free time, this proved to be highly motivational and made the mathematical content more exciting. The feature and its “story” involving mathematical content served as an effective conversation starter for the students’ discussion. It can already be reported that the method is currently being adjusted for the second cycle of the design research by designing the testing more as
a laboratory situation and less of a whole teaching experiment. In that way, it is possible to research more about the individual processes of the speech development of the children. Still, it can already be stated that nearly every student was able to correctly repeat what he or she had heard. If anything was unclear, it was easy to repeat a certain part of the radio feature individually or in front of the whole class. In this way, auditory material can counteract transitory learning through possible repetition. Through the combination of listening, repetition (if needed) and conversation, the use of radio features successfully aided the development of lexical memory and was helpful when introducing new terms. In the beginning of the unit, the students were only able to explain abstract terms such as “fair” with help of examples:

Teacher: So what actually is fair?
Student 1: Uhm fair is.. mh.. when you say for example if two uhm for example one gets a gummi bear and the other one not then they find this unfair and fair is if the other one also gets a gummi bear.

This explanation could be considered a form of everyday register, as the speaker uses an everyday example. However, this changed throughout the four hours of the project. During the final presentation of their investigated games, the students were allowed to look at their worksheet and their written answers for support. Interestingly, they did in fact use the new mathematical terms and phrases while arguing about the game’s fairness.

Student 1: Uhm it’s unfair because uhm the game is unfair because there are eight of blue, one of red, red of two uhm.. red two of them.
Student 2: No there’s only one of red.
Student 1: Yes. Two of green and four of orange. Because you have more chances with blue and the others have less.. (reads the next question) If the game is unfair, what would you have to do to make it fair? You’d have to change the game so that everybody had equal amounts of uhm cards of every colour then everyone would have equal chances and the game would be un uhm fair.

Teacher: Very good, thank you.

Here we see that the speaker uses school as well as technical register and also gives a causal argumentation. Mathematical terms and phrases, such as, “It’s unfair because...”, “more chances”, “equal amounts of...” and “equal chances” are not only used, but also used correctly. Referring to the MRA model explained above, there has been a development from the stage “Noticing” – where new terms or expressions are introduced – to the second stage “Intake” – which describes the process of understanding. There are also first elements of the third stage “Integration”, as there is testing, feedback and modification in the investigation of the game’s fairness. To reach the last stage “Output”, the children would need to be able to use the new terms fluently without support. In this case, the children
still have the support of their written answers on their worksheets while presenting their results, so we are unable to tell for sure if they reached this stage. However, it is clear that there is a development from stage one to three – from noticing to integration. This indicates that there is an apparent improvement in the students’ mathematical expression throughout the teaching unit. Children are offered professional language and are thus challenged to intake and use it. The absence of visuals, gestures and deictics in auditory material (unlike YouTube videos etc.) is a big challenge and opportunity for language development which worked out in my pilot testing quite well. Thus, as a first conclusion, it can be stated that radio features or auditory material in general can indeed serve as verbal language support in mathematics education. The goal of my main study is to verify and specify this statement for other mathematical topics, to further research on the use of the features in laboratory situations and to develop various possible applications of radio resources as auditory learning material.

References


STUDENTS’ VISUAL REPRESENTATION OF FRACTIONS AND EXPONENTIATION

Juan Luis Piñeiro, Olive Chapman, Elena Castro-Rodríguez and Enrique Castro

Abstract

Problem solving, as an important process in learning and doing mathematics, requires ongoing attention in research on teachers of mathematics in order to better support their development of appropriate knowledge to meaningfully engage students in it. This paper discusses an instrument to explore prospective primary school teachers’ mathematical problem-solving knowledge. Specifically, we provide an overview of the theoretical perspective framing a questionnaire, the process of creating and validating it, key items of it, and preliminary findings of prospective primary teachers’ knowledge obtained from it.

Keywords: problem solving, prospective primary teachers, PS questionnaire

Introduction

Problem solving [PS] is not only central to learning and doing mathematics but is also considered to be one of the 21st century competencies needed to deal with real-world challenges. Helping students to become proficient problem solvers in mathematics is also helping them to develop a way of thinking to apply in solving real-world problems. To achieve this, teachers need to hold PS knowledge for teaching (Chapman, 2015; Foster, Wake and Swan, 2014). Thus, this knowledge

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is an essential part of the knowledge prospective teachers need to develop in their teacher preparation programs. Research that explores the extent to which prospective teachers hold this knowledge is important to inform teacher education in order to provide them with appropriate learning opportunities to support their students’ learning of PS. Such research requires instruments or techniques specifically focused on PS (Foster et al., 2014) to address limitations in using general approaches intended to determine teachers’ professional proficiency (e.g., Flores, Escudero and Carrillo, 2013). This paper reports on the development of such an instrument (a questionnaire) used in a study to explore prospective primary school teachers’ knowledge of PS at the beginning and end of their teacher education program. While PS has been researched extensively over the years (Chapman, 2015; Schoenfeld, 1992), there has been more focus on problem solvers and less attention on the teacher, particularly, what PS looks like from the perspective of teachers’ knowledge (Lester, 2013), to which this study contributes.

**Theoretical Perspectives**

Genuine mathematical PS has been defined as a process and way of thinking (e.g., Chapman, 2015; Lester, 2013; Mason, Burton and Stacey, 2010; Schoenfeld, 1992) aimed at finding solutions to “something or some situation … when someone experiences a state of problematicity, takes on the task of making sense of the situation, and engages in some sense-making activity” (Mason, 2016, p. 263). We adopt this perspective of PS as an action taken by an individual or a group, who identifies a task with no direct procedure to solve it, proceeds to solve it by deploying a strategy involving a series of not necessarily linear steps and confronts the challenge with a favourable disposition (Chapman, 2015; Kilpatrick, Swafford and Findell, 2001).

The theoretical bases for framing the questionnaire consisted of the following: (1) We adopted the PS knowledge for teaching framework (Chapman, 2015), which consists of PS proficiency and “PS content knowledge … [consisting of] knowledge of problems, PS, and problem posing; Pedagogical PS knowledge … [consisting of] knowledge of students as problem solvers, instructional practices for PS; [and] Affective factors and beliefs” (p. 32). We used relevant sections of this framework to analyse curricular guidelines to identify the knowledge required to teach PS (Piñeiro, Castro-Rodríguez and Castro, 2016). This resulted in minor modifications to the framework to satisfy these requirements (Piñeiro, Castro-Rodríguez and Castro, 2019). (2) We also drew on mathematical competence theories (e.g., Kilpatrick et al., 2001) and PS proficiency theories (e.g., Chapman, 2015) to establish a characterization about PS. (3) Finally, we applied the didactic triangle (Schoenfeld, 2012), that is, the relationship between teacher, student, and content, to interpret pedagogical knowledge related to PS.
These perspectives resulted in two themes used to determine and categorize items of the questionnaires: teachers’ knowledge about PS and teachers’ pedagogical knowledge of PS, with their respective components as follows.

**Teachers’ knowledge about PS.** Table 1 provides a summary of this knowledge consisting of three components: problem characterization (i.e., whether a task is a problem), PS process, and disposition (i.e., the involvement that is generated when realizing a truly problematic task) (Piñeiro et al., 2019).

<table>
<thead>
<tr>
<th>Component</th>
<th>Knowledge</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem characterization</td>
<td>Task with no known solving procedure</td>
</tr>
<tr>
<td></td>
<td>Problem solver’s consideration</td>
</tr>
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<td></td>
<td>Type of tasks posed as problems</td>
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<tr>
<td>PS process</td>
<td>PS stages and their characterization</td>
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<td></td>
<td>Strategies</td>
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<tr>
<td></td>
<td>Metacognition</td>
</tr>
<tr>
<td></td>
<td>Non-cognitive factors</td>
</tr>
<tr>
<td>Disposition</td>
<td>Acceptance of the PS challenge</td>
</tr>
</tbody>
</table>

Table 1: Components of teacher's knowledge of PS

**Teachers’ pedagogical knowledge about PS.** Table 2 provides a summary of this knowledge that consists of four components: student as problem solver; problems and PS as a school task; non-cognitive factors; and instructional approaches for PS (Piñeiro et al., 2019).

<table>
<thead>
<tr>
<th>Knowledge</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student as problem solver</td>
</tr>
<tr>
<td>Characteristics of successful problem solvers</td>
</tr>
<tr>
<td>Difficulties and mistakes</td>
</tr>
<tr>
<td>PS as a school task</td>
</tr>
<tr>
<td>Problem selection</td>
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<tr>
<td>PS models and strategies</td>
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<tr>
<td>Problem posing</td>
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<tr>
<td>Non-cognitive factors that affect PS</td>
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<tr>
<td>Beliefs and conceptions influence on PS</td>
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<tr>
<td>PS instruction</td>
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<tr>
<td>Teaching approaches to PS</td>
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<tr>
<td>Discourse</td>
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<tr>
<td>Blockage</td>
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<tr>
<td>PS assessment</td>
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<tr>
<td>Resources</td>
</tr>
</tbody>
</table>

Table 2: Components of teacher’s pedagogical knowledge of PS

**Research Process**

The research process involved developing and testing an instrument for exploring teachers’ PS content and pedagogical knowledge based on the theoretical components in Tables 1 and 2.
Instrument development

A questionnaire format was adopted for the instrument. Such instruments have the power to collect information, among other things, in order to describe the knowledge of a sample of people (Fink, 2003). A closed questionnaire format was used since the goal was not to identify the meaning teachers held of problems, PS, and their teaching of it. Instead, the goal was to characterize their knowledge based on obtaining certain responses. This led to the use of dichotomous responses that indicate the presence or absence of certain knowledge (Fink, 2003).

The development of the questionnaires consisted of the following consecutive stages: (a) theoretical analysis of PS competence; (b) study of PS curricular requirements in primary education in Spain; (c) review of research of PS with primary school teachers; (d) development of the pilot questionnaire; (e) expert evaluation and testing of pilot questionnaires and (f) refinement to a final version of the questionnaires.

The pilot instrument consisted of two questionnaires: a PS knowledge questionnaire associated with Table 1 and a PS pedagogical knowledge questionnaire associated with Table 2. In the stage c of the development, the researchers ensured that the content was relevant and consistent with the selection of knowledge related to PS based on the Primary school curricula (Piñeiro et al., 2016). We present details here regarding the last two stages (e and f) of the development.

Expert evaluation. The pilot questionnaires were reviewed by experts to provide a qualitative evaluation of the degree of adequacy of the items to determine the PTs’ knowledge. Their expert judgment was also considered to be an indication of the validity of the items. Five Spanish mathematics education experts conducted the process. They were selected based on the criteria proposed by Skjong and Wenworht (2000): (a) research experience, publications and projects; (b) recognition in the scientific community; (c) availability and motivation to participate in the process, and d) impartiality in the investigation process. They considered: (a) each of the proposed items of the questionnaire in terms of their relevance to assessing the teacher's knowledge about PS; (b) correspondence of each item to its dimension, component and knowledge (Tables 1 and 2); and (c) suggestions for improvements.

Pilot Testing. The pilot questionnaires were also tested to obtain information to increase reliability, validity and feasibility (Cohen, Manion and Morrison, 2011). This involved assessing, for example, the adequacy of total time to complete the questionnaires and clarity and comprehension of the questionnaire items/statements.

The participants in the testing were 19 fourth-year prospective primary school teachers (PPTs) enrolled at the University of Granada, Spain. They had completed the elective course on ‘Mathematical Skills in Primary Education’, which
contained lessons on PS. These lessons included a focus on strategies and heuristics, problem posing, and PS teaching strategies.

The questionnaires were administered to the participants in four sessions: two for the knowledge about PS questionnaire and two for the pedagogical knowledge about PS questionnaire. The time each participant took to complete the questionnaire in each session was recorded. At the beginning of administering the questionnaires, the instructions were read and discussed with the participants. In addition, the purpose of the study was made explicit to them. After completing the questionnaires, they were asked to indicate in writing any difficulties they encountered or perceived regarding the comprehension and wording of the questionnaires’ items/statements.

Results on the Expert Evaluation and Testing

This section addresses the results of the expert evaluation and the testing of the pilot questionnaires regarding PS knowledge and PS pedagogical knowledge.

Expert Evaluation

The experts’ evaluation provided strong support for the validity and reliability for both questionnaires. In the case of the PS knowledge questionnaire, there was only one item (an open question) that received low evaluation and was removed from the questionnaire. Other changes involved surface features; in particular, introductions were added to the questionnaire, the items were reorganized in each section to better group them on common themes, and the wording of several questions was modified (e.g., not using compound sentences, using more appropriate terminology, and numbering some items).

There was a similar situation for the PS pedagogical knowledge questionnaire. Changes consisted of adding an item regarding the use of PS strategies, deleting an item on problem posing, making some modifications of wording to avoid repetition and other revision to the writing, and adding introductions to the questionnaire and each section.

Pilot Testing

The pilot testing of the questionnaires served two purposes: to obtain feedback on the mechanics of administering them and determine reliability of them based on feedback of the outcome regarding the nature of the PPTs’ knowledge.

Feedback on mechanics. This provided information on adequacy of total time to complete the questionnaires and clarity and comprehension of the items. The findings indicated total time of one hour to respond to the questionnaires and resulted in important changes to improve the clarity and comprehension of the questionnaire items. For example, one item stated: “For a task to be a problem, it depends on: (a) Only the solver; (b) Only from the solver's experience; (c) Both.” All three cases required a response of yes (agreeing) or no (disagreeing). The PPTs experienced difficulties understanding the question since option (a) did not reflect
the solver’s level of development and option (c) led to confusion causing them to not answer (a) and (b). Another example requiring clarification was the item that involved identifying the problem solver’s consideration regarding the possible solver. In general, some of the substantial modifications made were related to the separation of statements, adding context of who solves a problem, and changing statements that contained denials.

Feedback on outcome. The outcome of the PPTs’ responses indicated that the questionnaires could provide meaningful, reliable information regarding their knowledge of PS. For example, the results regarding “PS knowledge” related to problem characterization indicated what they knew, what they seemed confused about and contradictions in their knowledge of PS. A significant outcome was that the instrument revealed contradictions they held in their knowledge about the role of procedure in identifying a task as a problem. For example, according to our theoretical perspective, a problem is a task for which there is no direct procedure previously known. However, on one hand, the PPTs disagreed that for a task to be a problem it must be solved with previously learned procedures and, on the other, they did not agree that a problem is a task without a known procedure. Similarly, regarding consideration of the problem solver, they agreed that the possible solver should be considered in labelling a task as a problem. However, they were only able to recognize the problem solver consideration in some cases that involved a problem scenario instead of a statement to respond to. There were also contradictions in their knowledge of the structure/types of tasks and features of tasks that make them problems. For example, they considered closed routine and non-routine tasks as problems, were less positive about open tasks and considered features such as more than one response or not having all necessary information as desirable for problems.

Final Instrument

The final instrument after modifications from the experts’ evaluation and pilot testing consists of the two questionnaires: (1) a PS knowledge questionnaire and (2) a PS pedagogical knowledge questionnaire. We provide a condensed version of them here. The first author could be contacted for the complete instrument.

(1) PS knowledge questionnaire. This consists of two main parts: (1a) problem characterization and (1b) PS process based on Table 1; dispositions are in both.

(1a) Problem characterization. This consists of three themes:

a) Problem based on procedure. This consists of three items related to whether a problem is dependent on knowledge of procedure: “Problems must be solved only with previously learned procedures”; “A problem is a task without a known procedure to solve it”; and “When solving a problem, the solver must have mathematical concepts that allow him to articulate a solution procedure.”

b) Problem based on problem solver. This theme consists of five items related to whether a problem is dependent on students’ consideration. For example:
For a task to be a problem, the student must be able to detect a possible way of solving it in the first moments of its approach.

To practice arithmetic operations in the second year of primary school, textbooks at the end of the lessons propose tasks such as the following: The day I turned 8, my family brought me 6 presents in the morning, and 5 more in the afternoon. How many gifts did I receive that day? These kinds of tasks are problems?

c) Problems based on types/structure. This theme consists of six items related to whether a problem is dependent on its classifications (i.e., as routine/non-routine, verbal or non-verbal problems and applied/not applied contextual problems: routine applied, routine not applied, non-routine applied open and closed, and non-routine not applied open and closed). These classifications are not indicated in the questionnaire. These items include: “How many sweets do your classroom eat in a week?” for non-routine applied open problem. This section also consists of nine items that address the features of the problem, for example, how the data is presented, multiple solutions, etc. The items include: “A problem should always consider a context that reflects a situation” and “A problem must have all the necessary data in the statement.”

(1b) PS process. This section of the questionnaire consists of four themes:

a) Stages of PS. This theme consists of 25 items regarding identifying and characterizing the PS stages. Fifteen of these items explore the participants’ knowledge of the stages of the PS process. For example: “When solving a classroom mathematics problem, students advance toward the solution without backtracking” or “Students solve problems step-by-step.” The others 10 items focus on the stages of understanding and looking back. For example: “Understanding a problem entails determining what information is available and how it is structured.” “Once a problem is solved, it’s advisable for the solver to know what would happen if some of the data were changed”.

b) Strategies of PS. This theme consists of eight items regarding participants’ ability to identify specific strategies in students’ hypothetical answers to the

![Figure 1: Example of a strategy item](image)
problems. These items were formulated as multiple-choice questions to determine the teachers’ knowledge to PS strategies primarily related to the school mathematics curricula, with options being building a table, work backwards, draw a diagram, guess and check, look for a pattern, and operating. Example for one item is: “A farmer was counting his ducks and sheep. He counted 10 heads and 26 feet in all. How many ducks and sheep does he have?” followed by a hypothetical answer and options as shown in Figure 1.

c) Metacognition. For the eight items on this theme, four focus on the role played by metacognition in the PS process and four on the identification and explanation of an error regarding monitoring in a possible student's response. For example: “An awareness of one’s own knowledge helps choose the most suitable way to solve a problem.” “When someone solves a problem, they perform mental exercises that reveal when they’ve made a mistake.”

d) Non-cognitive factors. This theme consists of four items regarding the role of non-cognitive factors, specifically motivation, in the PS process. For example: “A problem can be successfully solved even when there’s no motivation to do so.” For each of the two questions on dispositions related to teachers’ awareness of the role of willingness in PS, one was grouped in the section on characterization and the other in the section on PS process to address the relationship to them. These items are: “A problem is a task that the solver accepts as a challenge.” “It’s important to know how to solve problems, but it’s more important to want to.”

(2) PS pedagogical knowledge questionnaire. This questionnaire (the second of the two) is divided into two parts: (2a) PS learning and (2b) PS teaching. The PS learning part was designed based on the first three knowledge of Table 2.

(2a) PS learning. This part is organized around three themes:

a) Student as a problem solver. The focus here is to identify the knowledge that teachers have about the characteristics indicated in the literature for successful problem solvers and unsuccessful or novice problem solvers. The theme consists of 15 items, for example: “They are persistent in maintaining the selected strategy has plan.” “They have little clarity of the way forward to reach the solution.”

b) PS as worthwhile task. This theme explores three ideas: (i) Good problem features reported by literature; addressed with seven questions. For example, “They allow exploring and developing mathematical ideas.” (ii) Knowledge of strategies and possible use by the teachers in classrooms, and PS process representation, that is, whether teachers conceptualize the PS process as a cyclical or linear. This is addressed with 38 items. For example: “[A cyclical diagram of PS] represents the PS process in a real way because it shows that you can go back over what has been done or skip phases.” (iii) The benefits and characteristics of problem posing; addressed with nine questions, including “Problem posing can encourage the use of wrong strategies.”
c) Non-cognitive factors that affect PS. This theme consists of eleven items that explore some of the most common beliefs about PS and how they mutually affect teacher and student when PS is taught. For example: “Students must solve problems as quickly as possible.” “Students should only solve problems once the mathematical concept has been taught.”

(2b) PS teaching. This part of the questionnaire is organized around 5 themes:

a) Teaching approaches to PS. This theme explores the goals of each of three teaching approaches to PS with three items and some of their characteristics with nine items. The three items include: “Teach mathematical concepts first, and then apply them to solve problems.” The nine items consist of examples of classroom situations that are related to some of the approaches and include statements as: “There must be an environment in the classroom where it is possible to explore problems both individually and in groups, communicating all the multiple ways of solving them.” “PS’s phases and strategies must be taught directly and explicitly.”

b) Discourse. For this theme, actions related to discussion management and ways to conduct the PS process are explored by means of 10 items; for example: “Guide the discussion on how the problem was solved, what procedure was used.” “Encourage students to indicate their agreement or disagreement with the solutions of their classmates, giving justified reasons.”

c) Blockage. This theme consists of eight items to explore teachers’ possible actions when students have difficulties in solving a problem. They focus on difficulties with understanding and carrying out the plan and include: “If they made a mistake in a calculation, ask them to read the problem again until he understands it.” “Teachers need to identify if the error is on the understanding of the problem conditions or on strategy execution.”

c) Assessment. This theme deals with possible criteria and instruments to assess the PS process. Criteria are addressed with 16 items; for example, “The ability to identify keywords (take away, lost, etc.).” “The existence of productive attitudes and beliefs for PS”. Instruments are addressed with 7 items consisting of a list assessment instruments; for example, “Multiple-choice assessment” and “Personal interviews”, asking for the most suitable to assess PS proficiency.

e) Resources. This theme addresses representations and their roles in solving problems with ten items and manipulative materials with three items. These items seek to investigate the importance and possible use of representations and manipulatives in the classroom when PS is taught. The items include: “Promote the use of a single type of representation to avoid confusion.” “Use only formal or symbolic representations.” “It is not necessary for students to use manipulative materials; it would be better instead to teach them the mathematical symbols.”
Conclusions

The instrument (questionnaire) discussed in this paper has the potential to capture specific aspects of prospective primary school teachers’ PS knowledge for teaching proposed in previous research (e.g., Chapman, 2015). Expert analyses validated the appropriateness of the instrument to explore PPTs’ PS knowledge and PS pedagogical knowledge. The pilot testing of it resulted in modifications, specifically to eliminate ambiguities, to improve its reliability. The outcome of the pilot testing also indicated that it is reliable to provide meaningful information about PPTs’ knowledge regarding what they know and limitations in what they know to inform teacher education. However, the instrument is not intended to be used to exhaustively explore PPTs’ knowledge to teach PS but to focus on key aspects regarding knowledge of problems and the PS process. It is also not intended to be used for the assessment of the PPTs’ PS knowledge. Instead it can provide insights into PPTs knowledge in the context of research or teacher education that are useful to understand what they know that can form a basis to further develop their PS knowledge. Thus, it can be used by educators to understand their students and to further investigate and extend it to design a more comprehensive and meaningful research instrument.

Acknowledgement: This study is supported by Chilean PhD scholarship folio 72170314 and Spanish National R&D Project EDU2015-70565-P.

References


MIDDLE SCHOOL STUDENTS’ DIFFICULTIES IN PROBLEM SOLVING AND THEIR ROOTS IN ELEMENTARY EDUCATION: WHAT WENT WRONG FOR THEM?

Elena Polotskaia, Annie Savard, Alexander Calvacante and Osnat Fellus

Abstract

Problem solving is identified as key in learning mathematics. However, problem solving still remains a source of challenges and difficulties to many students. The purpose of this research is to turn attention to learned ways of solving problems that preclude students’ ability to adopt appropriate strategies in problem solving. To do so, we draw on Davydov’s work to distinguish between different quantitative relationships; Brousseau’s concept of didactical milieu; and work on metacognition defined here as decisions about steps and strategies in problem solving. Using these three lines of theories together, we analyzed data working with first-year, middle school students who failed their elementary school examination. Findings suggest that students rely on operational approaches emanate from their didactical milieu, which precludes the use of appropriate metacognitive tools when solving problems.

Keywords: problem solving, mathematics, elementary school, relational paradigm

Introduction

Mathematics curricula in Canada (e.g., Ministère de l’éducation du Québec, 2001) emphasize the importance of problem solving defining it as a goal as well as the means to teaching mathematics. In this context, a student’s task in mathematics...
learning is not to learn rote facts or memorize established strategies, but to engage in analysis and reasoning processes when solving problems. In elementary school, problems are, more often than not, solved using arithmetic, i.e., concrete numbers and operations on them, while in middle and secondary schools, students are expected to use algebra, i.e., equations, inequalities, functions, knowledge of structures and relationships (e.g., Bednarz and Janvier, 1996). In this paper, we turn attention to the arithmetic-algebra transition, where problems become mathematically more complex, and students who have not developed more sophisticated arithmetic strategies as they continue to rely on concrete numerical data and on sequential operations, experience substantial difficulties.

In discussions about students’ difficulties in learning mathematics, researchers focus on various factors that include cognitive and metacognitive processes, mathematical content, lesson and curriculum design, teachers’ proficiency, and ways of intervention. However, less research is devoted to exploring the association between students’ difficulties and their previous learning experience (Lemoyne and Lessard, 2003). In our previous work with elementary school teachers, we observed that teachers often blame students’ difficulties in their classes on the students’ under-preparedness.

In our research, involving first-year middle school students (12-13 year olds), we worked with students who failed the final elementary school examination. Evidently, we did not have access to the students’ previous learning experiences and we could not make a direct link between a student’s failure of a mathematical task and his/her particular learning experience. However, building on previous research, we constructed a solid hypothesis of the missing elements in these students’ knowledge that generated their difficulties in solving problems.

In this paper, we address the following questions regarding students’ unsuccessful approaches to solving an algebraic problem: What can be identified as missing knowledge? What mathematical concepts or learnt ways-of-solving problems might prevent students from adopting an appropriate strategy? Where do these knowledge gaps originate? We hope that the theoretical analysis described below can spark a conversation within the elementary school mathematic education community in regard to these questions.

**Theoretical framework**

**Concept of quantitative relationship**

It is well established in our domain that solving a word problem (arithmetic or algebraic) requires the student to grasp and analyse relationships between quantities that can be known or unknown. Thompson (1993) identifies this process as quantitative reasoning or reasoning about quantitative relationships. Davydov (2008) understood *quantitative relationship* as a key mathematical concept. Yet the expression *quantitative relationship* is often used in the literature interchangeably with relational expression used in a word problem. We explain
the difference hereafter. Regarding a description—Yan has 2 sweets more than John—the expression “2 sweets more” is usually identified as a quantitative relationship between Yan’s sweets and John’s sweets. In Davydov’s terms, this description presents a quantitative relationship of *additive comparison* consisting of three elements related to each other: The quantity of Yan’s sweets, the quantity of John’s sweets, and the difference between these two quantities (two sweets). Therefore, an additive comparison relationship is not just an expression of comparison, but a structure of three quantities each of which has a particular role vis-à-vis two others. Consider the description—Yan and John have 8 sweets together—where the term relationship is usually not used. In Davydov’s terms, this description presents a quantitative relationship of *additive composition* consisting of three elements: Yan’s sweets, John’s sweets, and the total of sweets (8). Drawing on Davydov’s (2008) theory of developmental instruction, we consider several types of such relationships. Table 1 presents some of these relationships, their representations, and example problems.

<table>
<thead>
<tr>
<th>Relationship</th>
<th>Representation</th>
<th>Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additive composition</td>
<td><img src="chart" alt="Additive Composition Diagram" /></td>
<td>Yan and John have 8 sweets together. How many does Yan have if John has 5 sweets?</td>
</tr>
<tr>
<td>Additive comparison</td>
<td><img src="chart" alt="Additive Comparison Diagram" /></td>
<td>Yan has 8 sweets. This is 5 more than the quantity John has. How many sweets does John have?</td>
</tr>
<tr>
<td>Measurement</td>
<td><img src="chart" alt="Measurement Diagram" /></td>
<td>Yan has 8 sweets. This is 4 times less than the number of sweets John has. How many sweets does John have?</td>
</tr>
<tr>
<td>Cartesian product</td>
<td><img src="chart" alt="Cartesian Product Diagram" /></td>
<td>The area of a rectangular wall is 48m². What is its length if its height is 6m?</td>
</tr>
</tbody>
</table>

**Table 1: Some quantitative relationships**
The notion of quantitative relationship, in Davydov’s terms, adds more clarity in our analysis of word problems and students’ reasoning processes in solving problems. Thinking in terms of isolated quantities or numerical values does not allow for a holistic relational understanding of a problem. Therefore, the simple relationships we described above present elementary logical units—building blocks to construct a relational holistic understanding of a situation. In traditional teaching (not based on Davydov’s Theory of Developmental Instruction), teachers, usually, do not discuss quantitative relationships with students in spite of the fact that a correct solution of more complex problem requires a relational understanding of the problem. This may mean that successful problem solvers grasp quantitative relationships mentally, but often do not express them explicitly. Our previous research (Polotskaia and Savard, 2018) shows that making these relationships visible by using visual representations and discussing them explicitly with students positively affects students’ success in problem-solving.

**The didactical milieu and students’ perceptions**

If we analyze the situation of solving a problem from the point of view of the theory of didactical situations (Brousseau, 1998), we should consider the problem-solving task as a didactical milieu created by the teacher for the student. According to Brousseau (1998), the milieu is composed of objects (physical, cultural, social, and human) and the subject interacts with these objects within the situation. From the teacher’s point of view, a word problem task might present sentences (words) in a natural language describing a physical world situation as well as some quantities (numbers) and their mathematical relationships. It can also refer to some mathematical concepts. All these objects are cultural in nature and are usually “visible” for the teacher. We will refer to this milieu as “the teacher’s milieu.” From the student’s point of view, not all of these objects are visible. They can be “missing” if, for example, the student does not pay attention to them or if the student does not possess the knowledge required to “see” them. We will refer to this as “the student’s milieu.”

During the process of solving a word problem, students can modify their milieu. For example, they can represent the problem visually, thus adding new visual objects to the milieu. They can also invoke from their memory some ideas (cultural objects) or lose them from their “field of view,” thus entailing that the objects are no longer in the student’s milieu.

In our study, we create and provide word problems for which the teacher’s milieu includes quantitative relationships, arithmetic operations, natural numbers, and some descriptions of life situations. No physical objects are used (except for paper and pencil). We try to select contexts for our problems from – real-life situations, so that students can relate to them and easily interpret them. However, students may not know or only partially know the mathematical elements that signify objects of mathematical culture. The goal of problem solving is to allow students
to interact with the proposed milieu and therefore reconstruct their knowledge or construct some new knowledge of mathematical elements. It can be a conceptual knowledge (e.g., additive relationship) or the skills to analyse, represent, plan, and solve mathematical problems. The former refers to a cognitive domain-specific knowledge; the latter refers to the decision of what tools to use to navigate within the milieu, recognise the objects and their roles in the situation, manipulate these objects, and create new ones thus modifying the initial milieu.

**Cognition and metacognition**

In analysing students’ difficulties in mathematics, research distinguishes between cognitive and metacognitive aspects of thinking and learning. Cognition in problem solving can include mathematical concepts and mathematical procedures. Metacognition refers to the regulation of the solving and learning process (e.g., Brown, 1978). Metacognition can take the form of decisions about steps and strategies in problem solving, beliefs about the task and methods of solution, judgment about self-efficiency, or impressions of the task difficulty. Research on metacognition in mathematics education attracts substantial attention especially in analysis of students’ difficulties. Researchers suggest that metacognition is a strong predictor of learning (e.g., Veenman, Van Hout-Wolters and Afflerbach 2006).

The cognitive and metacognitive aspects are interdependent within the process of solving a problem. Metacognitive skills are task specific and their development relies on the development of task-specific knowledge (Bryce and Whitebread, 2012). For example, knowledge about quantitative relationships can help to plan a calculation strategy and to evaluate the result of a calculation. Some researchers (e.g., Focant, 2003; Land, 2004) believe that metacognitive knowledge can potentially compensate for missing cognitive knowledge. For example, the use of graphical representations or models might facilitate rethinking of the problem and finding a solution strategy. At the same time, Focant (2003) insists that if a student’s understanding of the goal of a problem is incorrect, this misunderstanding will unfold a completely incorrect metacognitive process. Thus, a cyclic organisation of the process of problem solving and regular re-evaluation of one’s understanding of the problem can potentially help to avoid impasses.

In our previous publications (e.g., Polotskaia and Savard, 2018), we proposed a cyclic organization of the problem-solving process, which is based on a particular mathematical knowledge – quantitative relationships and arithmetic operations. For easy reference, we suggest the acronym COMPLETE to frame the four-part cycle of solving a word problem. The acronym stands for:

- **Literal Comprehension** – Reading the text of the problem several times until the life situation is understood.

- **Modeling** – Representing quantitative relationships graphically until all of the relationships are visually represented.
- **Planning** – Based on the representation, developing a solution strategy and identifying arithmetic operations to calculate.

- **Evaluating and Testing** – Calculating and making sense of the numerical results that are compatible with the representation and the text of the problem.

It is critical to highlight the non-sequential use of these components within the problem solving process. For example, the comprehension and modeling can be alternated in a cyclic way many times before planning, or they can be repeated after evaluation and testing if necessary. The COMPLETE organisation of problem solving can potentially support students’ metacognitive reasoning and allow for an efficient analysis of more complex arithmetic and algebraic problems. We used the four elements of the COMPLETE cycle as conceptual tools to analyse and code the students’ performance during interviews.

**Method**

Our participants were twelve 12 to 13 year-old students from remedial classes in one middle school. These students were assigned to the remedial class because they have failed their elementary school final examination. In one-on-one interviews, we asked our participants to solve eight word problems of various levels of complexity. In each problem, each numerical data was replaced with a blank space. We invited students first to solve the problem in a general form explaining what should be done to find the answer and formulate the arithmetic operation(s). We then provided numbers upon students’ request so that they could solve the problem in a traditional way. Students could ask for help in reading, and the researcher would read the problem as many times as requested. The interviews were video recorded. All talk was transcribed verbatim and all gestures were described prior to coding and analysis. Due to space limitations, we illustrate our analysis by giving one typical example of students’ work.

**Data**

Participants were given the following problem.

*The mayors of Laval and Sherbrook decide to plant _____ flowers in each of their cities. The two planting plots are rectangular and have the same area (length \times width). The dimensions of the Laval plot are ___m \times____ m. That of the Sherbrook plot is _____m long. What is the width of the Sherbrook plot?*

1. The student reads the problem out loud saying “a number” for the blank spaces. He stops reading the question and notes that the two areas are equal.

2. Student: “If there were the same numbers here (first dimension for Laval) and here (first dimension for Sherbrook) it would be easy to solve.”

3. Researcher: “And if the numbers are not the same?”
4. Student: “It will come to the same. If the numbers are not the same, the area will not be the same.”

5. Researcher: “Let’s put numbers in the blanks. Which blanks you need numbers for?”

6. Student: “The number of flowers planted is not important. I need numbers for the other three blanks”

7. Researcher: (Plugs in the numbers 36, 15, and 60.)

8. Student: Points to the number 15 and says, “This can be a factor of 60. We need to multiply this one as well” (pointing to the 36).

9. Researcher: “Can you write down the operations, or the operation?”

10. Student: “I would do 15×... It is like a missing value here. 15 times 1 is 15, 15 times 2 is 30, 15 times 3 is 45, and 15 times 4 is 60. (Writes the expression 15×4=60). (Points to 36) “This one as well.” (Composes the expression starting from ×4) 36×4=. “It will be a bigger number.”

11. Researcher: “We can calculate it later. Are you sure that we need to do these operations?”


Table 2 presents our analysis of this excerpt using the elements of the theoretical frameworks discussed above as units of analysis. Specifically, students’ work was analysed using Davydov’s terms of relational thinking, Brousseau’s concept of milieu, and the metacognitive approach of the COMPLETE cycle. We organized the analysis by the three theoretical dimensions in the table below. The numbers in parentheses indicate the lines in the transcripts to which the analysis refers.

<table>
<thead>
<tr>
<th>Description</th>
<th>Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Relational thinking perspective</strong></td>
<td></td>
</tr>
<tr>
<td>(6) The student recognises the structure of the problem enough to identify the irrelevant data, and does this without any numerical information.</td>
<td>The student is familiar with problems having extraneous data.</td>
</tr>
<tr>
<td>(8-10) The student does not coordinate his solution with the whole mathematical structure of the problem. His understanding of the role of some numbers is incorrect.</td>
<td>The student does not master the quantitative relationships and their properties. The student’s mastery of the relationships between the quantities does not allow for treating a complex mathematical structure (multiple relationships).</td>
</tr>
</tbody>
</table>
### Didactical milieu perspective

| (1-2) Working with the text with blank spaces, the student’s milieu includes the knowledge about area of a rectangle and its equalities. But the equivalence of two areas implies for him the equality of dimensions. | Working with problems without numbers might enhance students’ relational thinking. The distinction between notions of area and shape is not well established. |
| (7-8) When the numbers appear in the problem, the milieu changes for the student. He speaks about calculation and not about relationships between quantitates in the problem. | The knowledge about relationships is not connected to or integrated in the knowledge about numbers and operations. |
| (8-10) The student’s experience with numbers and calculations is the first option the student opts for at the expense of relational thinking. | It is possible that the order of numbers in the text (36, 15, 60) evokes the association with the area calculation expression (36x15=60) so that the last number is associated with the “result” of the area calculation. It is also possible that the values (the fact that 15 is a factor of 60) induce the operational thinking in the student. It is also possible that the relatively big number 60 is associated with the value of an area just because it is bigger than the other two. |
| (10) The student’s milieu includes the knowledge of “missing value”, but without the knowledge about inversion of the operations. | The reversibility of the arithmetic operations is not adequately developed or not associated with a “missing value” situation. |

### Metacognitive perspective

| (1) The student does not pay attention to the question of the problem. His metacognitive process does not include the visual representation of the relationships. | The student has no habit in solving complex problems thus his metacognitive knowledge is not adequately developed to support the solving process. |
| (10) The process of solving, as well as its validation, is guided by the “missing value” concept. Preoccupation with lack of knowledge about relationships between quantities affects the formation of | |
arithmetic, i.e., calculation, replaces the goal of the problem, which highlights the relationship of the equality of two areas.

metacognitive orientation for the task. The orientation drifts from a relational approach to the operational approach.

Table 2: Analysis and teaching hypothesis

Discussion and conclusion

We analysed the performance in problem solving of students experiencing difficulties in mathematics trying to hypothesize about their previous learning experience and missing knowledge. The three perspectives of analysis—combined—offer a more comprehensive system of analysis that helps us to identify several important elements, which may not be otherwise clearly associated and consequently not adequately attended to in students’ experience of problem solving. The combined theoretical framework has generated the following insights.

First, knowledge about quantitative relationships is not represented in explicit guidelines of the elementary curriculum. The presence of numbers in the text transforms students’ milieu from employing a relational to using an arithmetic (calculation) one. Therefore, current word-problem activities tend to contribute more to the development of numerical knowledge than to relational thinking. Some students will develop the latter to some extent, but many others will certainly benefit from activities explicitly discussing quantitative relationships (by solving problems without numbers, for example). Our previous research (e.g., Polotskaia and Savard, 2018) in elementary school suggests that the transition from the calculation approach to the relational approach is feasible and can be managed by a teacher in a regular classroom setting.

Second, lack of knowledge about relationships and their visual representations negatively affects the metacognitive organization of the solving process, which often drifts toward some calculations and draws students’ attention to numerical values per se. In such contexts, the goal of respecting quantitative relationships in a problem disappears. It is replaced by “calculating a missing number,” and cannot serve as an evaluation and validation tool within the metacognitive solution cycle (Focant, 2003). Third, some concepts, such as area and shape, need a more thorough treatment in elementary classes to allow all students to develop a clear distinction between them and to more easily identity relationship between shapes and quantities.

Finally, we draw on the work of Davydov who, among other concepts related to the teaching of mathematics has also introduced the concept of quantitative relationship. Applying and extending Davydov’s ideas in our own research in this project and in previous ones, we suggest the pivotal role a transition from
operational thinking to relational thinking can play in the teaching of mathematics. This shift can be introduced at the elementary grade levels and in turn alleviate the cognitive and metacognitive difficulties students experience in middle and secondary school. We used the term milieu to highlight the difference in available ways of reasoning and solving word problems. An appropriate metacognitive knowledge can only be developed within a well designed milieu, one that includes the knowledge about relationships and that is based on complex word problems presenting multiple relationships. Drawing on these findings we encourage teacher educators to emphasize the added value in paying careful attention to the combination of relational approach to mathematics with didactical milieux, and work on metacognitive skills in order to understand what it is that precludes the use of appropriate strategies when solving problems.

References


In the current paper we would like to present one activity that is part of a special year-long extracurricular course for grade three students in mathematical problem solving, we designed, aimed at instilling inquiry learning and argumentative norms. The Course consisted of 28 meetings in the academic year. The paradigmatic activity, we present, is dealing with the area concept. The design of the activity relied on four principles: (a) Creating collaborative situations, (b) Socio-cognitive conflicts, (c) providing tools for checking hypotheses and (d) inviting students to reflect on their solutions. We describe how a problem solving task designed according to the above principles, promoted students’ understanding of the area concept. Our main focus in this study was to investigate if, and in what way, principled design is effective for instilling a problem solving culture and for promoting conceptual learning.

Keywords: task design, problem solving, argumentation, area concept

Introduction and Theoretical Background

The main goal of this study is to exemplify the role of design in instilling a problem-solving culture that might promote conceptual learning. We present a case study in which dyads and triads of third grade students participating in a special year-long extracurricular course cope with an activity designed to trigger productive peer argumentation that might lead to conceptual learning. As a theoretical introduction, a preliminary review of four issues will be presented: (1) problem-solving in mathematics classes (2) the role of task design to facilitate problem solving environment (3) the role of argumentation in fostering conceptual learning in mathematics and (4) the area concept.

Problem-solving in mathematics classrooms

Several decades ago, leading researchers in mathematics education discussed the promotion of mathematical problem solving skills as a central goal in mathematics education. Polya (1945/1957), then Schoenfeld (1985), were responsible for breakthroughs in conceptualizing mathematical problem solving and relating it to mathematical thinking. They opened a vast educational field – how to learn to solve problems and how to teach problem solving. At the time when Schoenfeld (1985) wrote Mathematical Problem Solving, the prestige of cognitive psychology among researchers in education began to decrease, and many researchers had already adopted socio-cultural approaches to elaborate models of
instruction. One of those models - Cognitive Apprenticeship, enabled Schoenfeld (1994) to model problem solving to students and to scaffold their problem solving until the students reached autonomy. To teach problem solving, Schoenfeld (1994) favored the creation of a classroom culture based on: (a) emphasis on reasoning processes rather than on results only; (b) communication - the classroom setting encouraged and supported oral and written mathematical communication, and students were encouraged to evaluate, question, and challenge each others’ ideas and work; and (c) reflection on mathematical practice.

Teaching problem solving is difficult, since the teacher must decide when to intervene, and how, while at the same time leaving the solution essentially for the students to discover. Added to this, the challenge is how to manage and nurture this process for each student, or group of students, in the class. As stated by Burkhardt (1988):

“The teacher will often be in the position, unusual for mathematics teachers and uncomfortable for many, of not knowing; to work well without knowing all the answers requires experience, confidence, and self-awareness”. (p. 18)

These difficulties are often too complicated However, as we will show in the next subsection, an appropriate task design can considerably reduce them.

**Task design**

Task design consists of decisions taken to afford and constrain human behavior in tasks to be undertaken. Task design has been specifically focused on talk to increase the likelihood that students engage in productive talk Andriessen and Schwarz (2009) introduced the idea of argumentative design:

“Argumentative design concerns the design, by a teacher, researcher, or educational professional, of collaborative situations in educational contexts in which participants take on productive argumentation, or the exploration of a dialogical space.” (p. 145)

Andriessen and Schwarz propose a list of factors to be considered to design a task for productive argumentation.

In the present study, we list four factors as design principles: The first design principle is creating **collaborative situations**: Collaborative classrooms activate processes based on students' discussion and active work with the course material or activities. Teachers who use collaborative learning approaches should view themselves not as expert transmitters of knowledge, but as expert **designers** of intellectual experiences for students. The second design principle is creating a situation of **socio-cognitive conflict**. Cognitive conflict is triggered by surprise, uncertainty, curiosity, perplexity, and also argumentation. When the newly assimilated information conflicts with previously formed mental structures, it may result in disequilibrium or a cognitive conflict (Piaget, 1973). Neo-Piagetians,
such as Mugny and Doise (1979), added to Piaget’s theory the important role of social interactions in confronting conflict. They referred to conflict in the context of social interactions, and labeled it a socio-cognitive conflict: the collective occurrence of contradictory claims or understanding in social interactions. The third design principle is providing tools for raising and checking hypotheses. These tools encourage students to engage in argumentation as their presence in the task affords confrontations between expectations and conflicting evidence obtained through the checking itself. (e.g., Hershkowitz and Schwarz, 1999; Howe, Tolmie, Duchak-Tanner and Rattay, 2000; Schwarz and Linchevski, 2007). The last design principle is inviting students to reflect on their solutions. This was what Pólya (1957) theoretically described as the fourth stage of his model - understand, plan, carry out the plan, and look back (reflect).

**Argumentation and conceptual learning in mathematics**

From a cognitive point of view, the process of generating an argument, individually or collectively, is the seeking for an explanation/justification, for a claim, or a conclusion. As such, argumentation encourages thinking and learning. In addition, engaging in an argumentation process activity, demands from a student to express her/his thoughts verbally, or in any other “language”, in an explicit and clear way. In addition, argumentation is often initiated to refute a position, or a claim, and as such deepens understanding of the problem space (Baker, 2003; Hershkowitz and Schwarz, 1999). Furthermore, the special structure of argumentation discourse that interweaves premises, conclusions, rebuttals, etc., improves the organization of knowledge (Means and Voss, 1996). The last, Argumentative formats are likely to reduce the cognitive load. When discussants raising different views adopt one (integrated) view on a rational basis, they consider that view as objective (Baker, 2003). Empirical studies have shown that forms that are dialectical – in the sense that different ideas are critically considered, and dialogical, have clearly been shown as leading to deep learning (Baker, 2003; Asterhan and Schwarz, 2009).

**The area concept**

In this paper we present a task dealing with the area concept- a concept learned in elementary school. Area as the space inside a figure and the conservation of area are fundamental aspects of conceptual understanding in geometry (Beattys and Maher, 1985; Hughes and Rogers, 1979; Piaget, Inhelder and Szeminska, 1981). Yet, researchers report about many difficulties, which students have, concerning the area concept. One of them, as described by Friedlander and Lappan (1987), deals with identifying the change in the area of a figure caused enlargement of its sides by a linear scale factor:

“The principle of area growth presents many cognitive difficulties…it requires the recognition of the (somewhat counter-intuitive) fact that the enlargement of
a figure by a linear scale factor of n will increase its area by a factor of N SQUARE” (p. 140).

Stavy and Tirosh (2000) claim that in a broad perspective this response could be viewed as an example to the general intuitive rule: “Same A → Same B”. We see in the same approach the intuitive rule mentioned above by Friedlander and Lappan as: “Same change in length → Same change in area”.

The Research Design Experiment

The course

Twenty third grade students from different schools were selected for this program on the basis of a letter of recommendation from their teachers at the end of the second grade. We chose students with good understanding of quantitative information and of geometrical and spatial thinking. The course consisted of 28 sessions throughout the academic year. The activities enriched and expanded the mathematics learned in school. The structure of the meetings was fairly consistent, for creating a suitable context for productive engagement (Wood, 1999). Each meeting was about 75 minutes long. For the first 15 – 20 minutes, the instructor facilitated a whole class discussion to create a shared understanding of the activity; then, for approximately 5 minutes, each student engaged in the task individually; during the following 45 minutes, students worked in dyads or triads, solving tasks collaboratively and writing a common justification on a worksheet; for the final 5 – 10 minutes, the instructor led a whole class discussion to summarize.

Research Goal

Our main research goal is to explore if and in what way, principled design is effective for promoting a problem solving culture and for promoting conceptual learning.

The birthday cake activity as a principal research tool

The activity consisted of two successive tasks to which the four design principles were applied (a) creation of collaborative learning situations, (b) stimulation of socio-cognitive conflict, (c) provision of tools for checking hypotheses, and (d) creating opportunity for reflection upon and evaluation of solutions. In Figure 1 we present a shorten version of the activity.

The goal of the first task is to find out the students’ understanding of the area concept and its measurements. We assumed that while the students might cope with regular “text book” problems, they will use their intuitive rules such as Same A → Same B which in this case is wrong, if the questions are taken from their daily life situations. For this reason we designed the second task.
The procedure

Each of the tasks was first done individually, then the student in the dyads or triads were asked to reach a mutual solution. Doing so they had to argue and convince each other so the first design principle was achieved. The fact that they were exposed to different solutions create naturally the socio-cognitive conflict. (The second design principle). They answered the second task and then they explore the influence of enlargement of a side of square by a linear scale factor of n on its area using dynamic geometry tool (Geogebra) which served as a tool for testing hypotheses (The third design principle). Finally, after exploring with the Geogebra tool, the students were asked: Did you change your mind about the price of the enlarged cake? Answering this question demand reflection on previous solutions and the forth design principle was achieved as well.

Methodology of collecting data

The work of individuals, pairs or groups on the activity was observed, videotaped, transcribed and analyzed. The written worksheets of all students (individuals, dyads and groups) were analyzed in both quantitative and qualitative methods.

Data Analysis and Findings

Data analysis of the activity – quantitative analysis

We analyzed 34 worksheets – the collaborative work of dyads and triads from three successive years. We found that in task 1c 76.5% of the dyads found the
correct solution of the “text book” formal question. The correct solution is 32 square units. (See Figure 2.)

![Figure 2: The distribution of the solutions of task 1c](image)

Yet almost 85% of the dyads used the intuitive role of: “Same A → Same B” when they determined the price of the enlarge cake and they argue that the price should be 3 times the original price, meaning: $15. The correct solution is of course 9 times the original price meaning $45 and only 3% of the population (one group) gave this answer. (See Figure 3.)

![Figure 3: The distribution of the solutions of task 2](image)

Furthermore 81% of those who gave the correct answer to task 1c thought that the price of the enlarged cake should be $15. These findings indicate that when the environment is not “mathematically” intuitive rules such as Same A → Same B, outweighed the formal knowledge.
It is worth noting that after exploring the influence of enlargement of a side of square by a linear scale on its area with the dynamic geometry tool all students wrote the correct price of the enlarged cake which is $45.

In the next section we will present some of the reasoning processes we reveal while analyzing the videos and the worksheets.

**Data analysis of the activity – qualitative analysis**

**Tasks 1b and 1c**

As mentioned above 76.5% of the students gave correct answers to task 1c. Most of them drew the square units answering task 1b and then used the same manipulation to answer task 1c

Drawing the square units ensured the students about the correct solution which is four times the area of the original rectangle. (See Figure 4.)

**Task 2: What is the price of the enlarged cake?**

1. Interviewer: What do you think should be the price of the enlarged cake?
2. Romy: I think that the price is not $15 as Danny said, because he said that he wants a square cake in which each side is longer!
3. Ayelet: $15 and once more…yes I agree $30.

**The price of the cake after the use of Geogebra software**

6. Interviewer: The original price of the cake was $5 what should be the enlarged cake’s price be?
7. Ayelet: [draws the following and point to the segments] Here one and here one, here one …so it becomes nine times.
8. Romy: So the price of the enlarged cake should be 45$.
9. Interviewer: This is the original cake:  
10. Ayelet: So it is like that:  
11. Romy: Each side should be three times longer [sketch 1] [then she complete the drawing sketch 2]

12. Ayelet: If he says all sides it means from all directions. It should be from all directions around the original square.

So the price should be $45.
13. Interviewer: And if we multiply the length of each side by 5?
14. Romy: By 10?
15. Ayelet: In my opinion the area will multiply by 20 …25!
16. Romy: [draws the following drawing and said]
Yes you’re right it is 25 times the original square.

Discussion
In the course from which we focused on one activity, problem-solving was a context rather than a set of skills to be acquired. This context included the iterated enactment of practices such as small group work, oral and written reporting, and teacher-led discussion. In addition, the teacher encouraged students to focus on processes rather than on results, and to account for reasoning processes. We claim that the findings suggest that the meticulous design as well as the problem solving culture triggered a general process according to which students capitalized on problem solving heuristics and engaged in productive argumentation processes, to eventually reach deep understanding of a geometrical property: the fact that the enlargement of a square by a linear scale factor of \( n \) will increase its area by a factor of \( n^2 \).

The activity we described, indeed encouraged students to collaborate they were asked specifically to work in pairs or triads and to reach an agreement about their solutions. Collaboration led students to compare solutions. Since they were requested to justify their solutions, these justifications naturally created socio-cognitive conflicts. As we show in the qualitative examples the students used the
drawing as a heuristic to help them justify their solutions. Drawing a diagram was one of the heuristics they practice during the course.

The use of geogebra a DG software served as a tool for checking hypotheses (Andriessen and Schwarz, 2009; Hadas, Hershkowitz and Schwarz, 2001; Prusak, Hershkowitz and Schwarz, 2012). We show the impact of using this tool when at the end all students wrote individually the correct price for the enlarged cake.

To sum up, we show the importance of the design, and of the culture of problem solving which led students to engage in productive argumentation and as a result gain a meaningful understanding of the area concept. We believe that using learning environment as described in this paper and using activities that were designed according to the design principals we presented, may lead to inquiry and significant learning in math classes.

References


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**STRATEGIES USED BY FIRST GRADE PRIMARY SCHOOL STUDENTS DURING THE WORK ON A GEOMETRIC TASK CONCERNING ARRANGING THE PLANE**

* Marta Pytlak

**Abstract**

The importance of geometry not only in mathematics education, but also in general education is often discussed. The open question remains, how to exploit the potential and power of geometry in the development of students’ mathematical thinking. In this paper the preliminary results of a research carried out among students from the first grade of primary school will be presented. The study shows how students deal with the arrangement of the planes.

**Keywords:** early geometry, creative and critical thinking, geometrical patterns, regularity

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**Introduction and theoretical framework**

Mathematics education at the lower levels of education is increasingly being discussed. It emphasizes the importance of adequate preparation of the student not only at the stage of early school, but also at preschool. We can find references which are devoted to a discussion on developing intuition needed to carry out specific mathematical problems (see Ginsburg, 2004; Gruszczyk-Kołczyńska, 2009; Rożek, 2016; Schoenfeld, 1992). Much attention is given to the theme of the development of students’ mathematical thinking (Klus-Stańska and Kalinowska, 2005; Gruszczyk-Kołczyńska and Zielińska, 2009). In this respect, geometry plays quite an important role.

Geometry is an integral part of our daily life even if we do not realize that. Geometry teaches us the basic skills of logical thinking and reasoning. We can observe that in the latest trends in the early education a large emphasis is placed on developing the skills needed for a child to explore and understand the world, to cope with different situations of their everyday life. The skills that are particularly useful in various situations include analyzing, critical thinking, putting and verifying hypotheses. The tasks of the school according to the new curriculum (MEN, 2008) include the care that a child could acquire the knowledge and skills needed to understand the world and equipment that is essential for the child in mathematics skills in real-life and school situations and for solving the problems.

The most important skills obtained by the student in the course of general education in elementary school should be, inter alia, mathematical thinking, comprehension as the ability to use the basic tools of mathematics in his or her daily live and carrying out the elementary mathematical understandings.

At the higher level of education mathematics is presented as a structured and ordered formal knowledge, with specified branches of mathematics. This gives the ability to do an advanced work on both learning and developing mathematical knowledge, as well as developing formal (symbolic) mathematical language.

At the lower levels of education (especially in the lower grades of primary school) ready-made, formal knowledge from different fields of mathematics is not presented for students, but they are introduced into the world of arithmetic and geometry (Hejný and Jirotková, 2006). The world of arithmetic is ultimately structured, governed by clear rules. The individual records and symbols used in this world are equally read by all. The situation is different in the world of geometry. As Hejný and Jirotková (2006) write:

> The world of geometry is a community of individuals or small families and there is a large diversity in the linkages between them. From the didactic point of view, arithmetic is suitable for developing abilities systematically, and geometry is more suitable for abilities such as experimenting, discovering, concept creation, hypothesizing and creating mini-structures. (p. 394)
So geometry is a very good way to develop students‘ mathematical thinking. By analyzing the historical development of geometry, we can note that it was accompanied by a man in its activities since the dawn of time. It was the first scientific field within mathematics which was created by people. Its importance for the ancient world was great. In mathematics, it had an important position.

Initially the geometry was not only a theoretical science but appeared with the need and desire of man to arrange the space around him, to solve many practical problems – beginning from the construction, by traveling to ornamentation (Hejný, 1990). This historical trait also points the way for approaches to the teaching of school geometry: geometric knowledge is born in a specific way-through action. It is important here to gain experience, and practical problem solving.

The importance of geometry in the education of children and young people was the subject of discussion of many researchers. There is a belief that the geometry can promote the overall development of the mathematical competence of the child. Concerning the importance of geometry in teaching children and youth Swoboda writes:

… quasi-geometrical activities can develop widely understood children’s mathematical competence. On the one hand, since geometrical approach to mathematics is closer to children than arithmetical one, geometry can open doors to a world of mathematics. Geometrical cognition starts from a reflection upon the perceived phenomena and in this way correlates with the basic ways of learning among children. On the other hand, it gives a chance to develop such ways of thinking, that are typical for mathematical thinking. Skills like generalization, abstraction, perceiving relations, understanding rules are the base for this aim. Early geometry is in-between physical and abstract worlds. By this, it enables to mathematize this world.” (Swoboda, 2009, p.29)

Geometry has a very large potential for the development of mathematical thinking of students. However, it is not taken into consideration by school education. It consists of a number of factors. As Paszkiewicz, Łyko, Mamczur and Swoboda (2001) wrote:

Teaching geometry is conducted in a very limited number of lessons and so children do not realize why they are learning it. It is just part of mathematics without any references to reality. Children were taught abstract geometry items and they did not have the opportunity to learn the properties of the figures through manipulating and even if they had it, it was only apparent. Such a style of teaching geometry does not give any chance for problem solving teaching. It also does not help a child to make links between the problem, the procedure of solving it and the solution. (p. 86)

Teachers from early education level often avoid geometrical topics. For many of them mathematics education is not a strong point. They focus only on the training of arithmetic skills. The geometry is limited to familiarize the student with the basic figures (square, circle, triangle, rectangle), measure the length of the
sections, drawing the reflectional symmetry of objects. And all of it is made in a very static way.

To use the full capabilities of geometry in the education of children and young people, the attitude to teaching must be changed. Geometry was born from an act, from human needs for developing and structuring of space around them. Therefore, an essential element in the teaching of geometry should be an action. Swoboda (2001) writes about it:

> Action plays an important role in the formation of geometrical concept because there is always correlation between concept and the activity addressed to the concept. The object from the real world are perceived as the gestalt. The way of gathering information is perception, but after that the action with the object leads to the verbal description in their properties. (p. 151)

Therefore, an important didactic problem is to organize the activity during the mathematics classes in such a way as to encourage students to actually participate in the lessons and to give them a chance to creative thinking and exploring mathematics. This is especially important for students who are just starting their adventure with education. Games that pupils use in everyday life have the great potential in this respect. These include for example building structures with blocks, arranging puzzles, the continuation of patterns. These are activities well known and easy to apply by students.

Given the above facts, it seems that geometry is a very good way to develop students’ mathematical abilities. Especially students at lower levels of education (it means kindergarten and first and second grade of primary school education, 3-7 year old children), who still are not able to read and write. In addition, geometrical topics can be presented to students by referring to the daily activities, which they face around. It is enough to present these issues for students in a suitable way adopted to them.

The above reflections on the potential of geometry were an inspiration to develop and conduct a series of classes. In this paper some of them concerning arranging the plane and making patterns are presented.

**Methodology of research**

**The aim of the research**

The observed classes have a pilot character and they prepare the ground for a larger research project on the development of spatial imagination of pupils.

The main goal was to encourage students to be more active in mathematics and develop their mathematical thinking (especially such elements as analyzing, formulating hypotheses and verification, argumentation). In addition, I wanted to see if the students will use their previous experience during the further work. More precisely, the aim of the research, that I wanted to achieve, was to get answers for the following questions:
• What intuition do 6-7-year-old students have in the field of geometrical transformations (such as symmetry, translation)?

• How do students cope with the arrangement of the plane? What strategies will be dominant during their work?

• Will they be able to recognize patterns and to continue them?

• Will students be able to be creative and constructive in their work?

**Research group**

Data presented in this paper were collected during the first grade classes. It was a series of meetings, whose main goal was to develop students’ interests and aptitudes for math. These were the additional activities that took place after the scheduled lessons. They took place once a week and lasted one-hour lesson (45 minutes). The classes were not compulsory and only eager students attended. Twenty out of twenty-three students from the first grade of primary school (6-7-years-old) took part in the sessions. The classes were conducted in March and April 2016. All lessons were recorded by video camera, and in addition, the work performed by a student was photographed.

The topics addressed in these sessions related to geometry. Through play students became familiar to concepts such as the arrangement of the plane, symmetries, translations and searching for regularity, continuing the pattern. Students participating in the classes have not met previously with raised geometric issues. These were children under the new educational reform, which was carried out in Poland, started primary school a year earlier (as 6, not 7-year-olds). Previously, these students attended kindergarten, where they were only familiar with the basic geometrical shapes such as circle, square, rectangle and triangle.

**Organization of research**

The whole cycle of activities consisted of three stages. On each stage students had to solve similar tasks, but the way of their presentation and execution was different.

All three stages had a common title *We help Winnie the Pooh to renew the house*. The stages were carried out in the convention of narrative stories, in which students actively participated. At the beginning the teacher presented to students a teddy bear and talked about his problems with the house. She asked the children on behalf of the teddy bear for help in laying the floor. The task that has been placed to the students was: “Winnie the Pooh wants to repair his house. He would like to renew the floor. Please, help Winnie the Pooh to arrange the floor, the most nicely as you can.” During the first stage students had access to square tiles, and their task was to arrange some pattern. Initially, they received cartons only in one colour. After laying a single coloured floor they received additional tiles in different colours (there were maximum eight different colours). After each puzzle
a conversation with the students about their work was carried out. The idea was that students had to present their way of work and tell about their inspiration during arranging the specified pattern. In the second stage, the students received a work sheet with the fragment of a pattern. Their task was to continue the pattern and complete the colouring according to the perceived scheme. Finally, they were given a blank sheet, on which they had to create their own designs.

The last stage concerned the joint work of students. The teacher arranged on the desk a fragment of the pattern and a selected student had to continue it. Then the students themselves had to invent patterns for others – one student arranged a fragment of the pattern and another student had to continue it. During this work the conversation about the ways of arranging was conducted all the time.

The results of students’ work

At each stage, students started the work very willingly. The research material consists of: students’ work sheets, a recorded video showing both students’ work and their conversation with the teacher. After analyzing all research materials, I was able to distinguish the strategies used by the students. Due to limited space in the paper, I present only a preliminary analysis of research. I focus mainly on the strategies used by students at each of the stages, especially during the first one.

Stage I

First of all three strategies of arranging appeared (both in the case of one or more colours of tiles):

- arranging from the top to the bottom
- arranging from the outside to the inside
- arranging from the left to the right

In the case of one colour of tiles the most popular pattern was a “chessboard”. Perhaps students associated this way of arranging with everyday life (the game board in chess or checkers). Students also tried to arrange other models, for example: crossing lines (they look like diagonal of rectangle) or rectangles. In each of them they tried to keep the symmetry.

When the students had more colours of tiles, in their work appeared mainly two types of patterns (about 95% of all works): chessboard and lines. The most frequently arranged pattern was a chessboard. However, the method of stacking was different than in the previous task. This time the students acted in two ways:

- they were planning a checkerboard pattern using one colour, as it was during the previous task, and then they filled the “gaps” using the second colour
- at the same time they rolled out alternative tiles in two colours, starting from the top of the sheet
In contrast to the situation with usage of only one colour of tiles, this time the students tried to fill whole sheet. Interestingly, they did not go out of the area. The teacher asked them how they arose the puzzle and if it is possible to continue it (except for the white card, e.g. on the whole plane of the desk). During describing the way of laying students usually showed how they were doing. Also they argued that it is impossible to further continue the arrangement, because there is the end of their work (the sheet has specific dimensions). Another obstacle for the continuation of the pattern was the lack of tiles. In such situation the teacher asked students “If I give you more tiles, what do you do? How will the pattern look like?” Some students responded “It is impossible to do anything further”. Then the teacher asked “Why?”. The students explained, that “to continue the pattern I need tiles (real objects), because I must see their colour. And after that I will be able to say how this pattern will be continued.” Such an approach shows the fact that students cannot think completely abstractly, that they need specific, real objects to act.

Another most frequently used motive was “lines”. Students were arranging from top to down.

In these works I could distinguish the following strategies of proceeding:

- arranging one colour lines, without a clear rhythm (using all available colour tiles)
- arranging lines according to a fixed rhythm (using only two or three colour tiles)

For students taking the first strategy, it was only important that they use all available colours tiles. Hence, they rolled out one colour lines, each in a different coherent. As a justification for their work they reported that they created the rainbow. We could see it clearly in the following works (Figure 1):

![Figure 1: Example of students’ work strategy “lines”](image)

Students who used the second strategy tried to apply a certain rhythm consistently. It concerned the same tiling or use different colours. Some students only used two colours, arranging them alternately (Figure 2):
A few students used the rhythm involving a specific colour sequence, e.g. yellow, green, pink, blue, yellow, etc. We can see this in the following works of students (Figure 3):

For many students it was a strong need to maintain symmetry in their work. This was particularly evident in these works, where students started laying tiles from the edge of sheet and gradually provided it to the inside. Symmetry was not only preserved in the way the arrangement of tiles, but also in the choice of colours.

**Stage II**

When filling out the work sheet from the second stage of research, students used two strategies:

- the whole presented a fragment of the pattern they treated as a “rubber stamp”, which was consistently referred as “a bounce”, to complete the sheet
- they have separated from the presented pattern a small fragment, which was subsequently continued

Definitely, the first way of working was dominant. The students looked on the presented pattern as a whole and tried to continue it. Sometimes students exactly copied the existing part (Figure 4, on the left). They even did not bother about the fact that there were empty spaces on the sheet. They treated it as a part of the pattern.

Sometimes there were some modifications. This happened in a situation when the “rubber stamp” was not distinguished as an ideal rectangle filling the sheet completely from right to left. Then the “gap” which appeared, was filled out by one’s own creativity.
The second way of working was less popular. It demanded from the student the ability of analyzing the pattern and isolating the smaller fragment, which then had to be consistently repeated. Such an approach required from the student an analysis of the relationship in the given pattern (Figure 5).

![Figure 5: Example of a student’s work](image)

**Stage III**

It was the same during the third stage. This time, the students had to continue the pattern, using tiles from the first stage. This time, most of the students treated the framed piece as a whole, which should then be reproduced. They extended the pattern only in one plane, according to one axis. Even at the suggestion of the teacher, to try to extend the pattern in two directions, they argued, it would not be good.

Only a few students noticed that the puzzle is created according to a fixed pattern, which can be freely repeated both vertically and horizontally. These students were able to distinguish this recurring segment in the whole pattern arranging by the teacher. Such an attitude might indicate a more analytical approach to the task.

**Conclusions**

Students taking part in the classes showed great creativity and ingenuity in solving their tasks. Although they worked spontaneously and were not in any way directed at certain phenomena, their work clearly showed the need for ordering and systematizing. The most emerging method of work was arranging from the outside of sheet to the inside or from top to bottom. Rhythm and arrangement were visible in almost every work. This was not too surprising because the result is similar to that obtained by Swoboda (2006).

The second stage of the research showed that for 7-8 year-old students it is easier to look at the task in a holistic way. They still have difficulties with detailed...
analyses. They can see the pattern and apply it, but it is rather the repetition of the whole fragment than the repetition of the part of the whole.

Students at this level of education need real objects to act. They are not able to go beyond concrete thinking yet. Abstract thinking with regard to geometry is difficult for them yet. They do not yet have the intuition of infinity with respect to geometric objects. Even the plane which is arranging by them must have “boundaries”. This is an important guideline for teachers, indicating the way of working with children at this stage of education. Geometry should be learnt through play, giving students as many opportunities to work on specific, real objects. This will allow them to develop appropriate intuition and will be big support in the further educational process.

Students presented different strategies and different attitude to the task. Even when the task seemed clear and obvious, the students behaved differently. This diversity was a good starting point for discussion. This, in turn, gave the possibility to raise critical thinking and argumentation.

Participation in the classes allowed students to gain new experiences relating to geometrical patterns and space arranging. They also gave them a chance to experience something new. These experiences can be used in further learning. According to Hejný’s theory of general model, it is very important to build our own mathematical knowledge (Hejný, 2004). The more experience, the better starting point for further learning. My goal is to see how the experiences gained during presented in this paper classes translate into next student development. This will be the subject of further, long-term studies.

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CONJECTURES AND JUSTIFICATIONS IN BUILDING TOWERS OF CUBES AND CUBOIDS – DESIGN OF A COMBINATORIAL LEARNING ENVIRONMENT

Christian Rütten and Stephanie Weskamp

Abstract

As a general competence in the German and international standards, reasoning can be fostered amongst others by working on combinatorial tasks. Conjecturing and its justification are crucial aspects of this competence. Primary students’ conjectures and justifications should be explored in the context of the combinatorial learning environment ‘building towers of cubes and cuboids’. This learning environment was designed in the project ‘Mathe-Spürnasen’ (‘good noses mathematics’) and further

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developed following a design research approach. Focusing on learners’ conjecturing and justifications, this paper presents the design process of the corresponding learning environment.

Keywords: substantial learning environments, design research, combinatorics, conjecturing, justification

Introduction: Conjecturing and justifications

As general competencies in mathematics, the development of conjecturing and its justification should be encouraged in all grades (cf. KMK 2005, Walther et al. 2011, p. 32). Such a “systematic reasoning is a defining feature of mathematics. Exploring, justifying, and using mathematical conjectures are common to all content areas and, with different levels of rigor, all grade levels” (NCTM, 2007, p. 11). As one content area to foster reasoning, combinatorics is suggested by Kütting (1981). Amongst others especially combinatorial problems provide a broad potentiality to evolve and increase such competencies in systematic reasoning (cf. Schipper, 2009; Neubert, 2003; Kapur, 1970). Normally, such problems require only little prior knowledge and complex calculating, and enable different modes of representation (cf. Schipper, 2009; Neubert, 2001). Therefore, combinatorial problems are often accessible to all learners and offer natural differentiation. In that sense, primary students can already develop their individual competencies in making conjectures and formulating justifications by solving combinatorial problems. In the project ‘Mathe-Spürnasen’ (‘good noses mathematics’, cf. Rütten and Scherer, 2015; Rütten, Scherer and Weskamp, 2018), different combinatorial substantial learning environments (SLE) for primary students (grade 4; 9- to 10-year-old students) were designed in the context of design research (see the section Framework of design research) to foster reasoning by conjecturing and its justification. For developing local theories, it is important to understand primary students’ conjectures and schemes of justification. Based on one of the combinatorial SLE designed in the project, these conjectures and justifications are explored.

Research questions

Solving combinatorial problems can be localized differently in mathematics education (combinatorial thinking, cf. Kütting, 1981, p. 173). Therefore, various combinatorial problems can be integrated already in primary mathematics. For example, combinatorial problems occur in the context of the Fibonacci sequence (cf. Rényi, 1982). In the project ‘Mathe-Spürnasen’ (see the section Embedment: Project ‘Mathe-Spürnasen’), combinatorial SLE in context of the SLE ‘Fibonacci sequence’ were designed and examined, whereby amongst others the following questions were guiding:

- What conjectures do learners make?
- How are these conjectures justified?
By these focuses, in the following, the design and research of one of these SLE will be described in more detail.

**Framework of design research**

*Embedment: Project ‘Mathe-Spürnasen’*

In the project, primary classes (grade 4; 9- to 10-year-old students) are offered a visit at university to explore a substantial mathematical topic from different perspectives by working with a SLE (cf. Rütten and Scherer, 2015). Every SLE consists of an introductory unit and three different follow-up units. Since each unit itself can also be characterized as a SLE, it seems appropriate to speak of learning environments within a learning environment (cf. Rütten et al., 2018). Understanding mathematic education as a design science, one central objective of the project is the design and research of SLE (cf. e. g. Wittmann, 1995).

**Learning environments within the framework of design science**

Wittmann (1995, p. 356 f.) describes the design and research of SLE as one specific task of mathematics education: “For developing mathematics education as a design science it is crucial to find ways how design on the one hand and empirical research on the other can be related to one another” (Wittmann, 1995, p. 364). For relating design and research, it is necessary to combine theory and practice. In this regard, SLE offer a broad potential for empirical research and provide insights into learning and teaching processes (cf. Wittmann, 1995, p. 368), which can vice versa be useful for the further design of SLE.

Methodological approaches for combining theory and practice, as described by Wittmann, can be identified internationally amongst others under the term *design research* (cf. van den Akker et al., 2006; McKenney and Reeves, 2012). Van den Akker et al. (2006) emphasize, that such “research incorporates a cyclic approach of design, evaluation and revision” (p. 4). Based on this, four working areas are mentioned in the context of the FUNKEN project: Structuring and specifying the learning object, (further) development of the design, carrying out and evaluating design experiments, and (further) development of local theories about teaching and learning processes (cf. Hußmann et al., 2013). In the section Concretization of design development, the iterative cycle of design and research as applied for the project ‘Mathe-Spürnasen’ (cf. Rütten et al., 2018) will be concretized by one of the follow-up units of the SLE ‘Fibonacci sequence’.

**A combinatorial SLE about ‘Fibonacci Sequence’**

*Concretization of design development*

The whole SLE ‘Fibonacci sequence’ with introductory unit and three follow-up units (geometrical: golden spiral, arithmetical: number chains, combinatorial: either pavement problem, building towers of Lego bricks or building towers of cubes and cuboids) was designed and tried out according to a design research
approach since 2012 (cf. Rütten et al., 2018). Regarding all three designed combinatorial follow-up units (see the section Building towers of cubes and cuboids), the focus was on the students’ learning processes with special consideration of the accessibility to the task, development of structuring strategies and the development of competencies about argumentation (cf. Rütten et al., 2018). The development of the different follow-up units took place successively. First, the so-called pavement problem was designed and tried out, then the problem of building towers of Lego bricks (see the section Building towers of cubes and cuboids). Finally, a SLE for building towers of cubes and cuboids was designed. Each of these units went through the design cycle (see the section Learning environments within the framework of design science). As result of design research, the SLE ‘Building Towers of Cubes and Cuboids’ is intended to offer an easier accessibility of the problem and of the underlying pattern of Fibonacci sequence, compared to the two other SLE. This is explored in the context of a further design cycle. First, the focus is on students’ argumentation, to generate local theories of students’ conjectures and justifications.

**Combinatorial structuring strategies, conjectures and justifications**

To develop such local theories within the context of design research, existing theories regarding solving strategies of learners in combinatorial problems as a basis for conjecturing and its justification as well as theories about learners’ justifications itself are examined first. English (1991) explored strategies of children aged between four and ten in solving combinatorial enumeration problems regarding the cardinal numeral of Cartesian product. Six solution strategies were identified, which English (1996) assigned to three stages of combinatorial structuring strategies: The first stage, called *nonplanning stage*, is characterized by “a trial-and-error approach to problem solution, selecting items in a random manner” (English, 1996, p. 94). English specifies the second stage, the *transitional stage*, by using underlying structures. This stage leads from less efficient procedures to the most efficient problem-solving strategy (*final stage*).

In addition to the model of English and Hoffmann (2003) differentiates between micro- and macro-strategies. While the term micro-strategy is used to describe patterns that are suitable for generating the following combination or combinations or parts of them, the term macro-strategy describes patterns that structure the whole set of figures (cf. Hoffmann, 2003, p. 34). Therefore, micro-strategies can be located on the transitional stage. Macro-strategies are characterized in the sense of more resp. of most efficient strategies that can potentially generate all possibilities (i.e. final stage). However, a distinction can be made between patterns that only potentially generate all possibilities and those that necessarily generate all possibilities. Structuring of the first type include macro-strategies that are composed of micro-strategies (e.g. problem solution in phases, complete inversion or complete counter-pair formation, cf. Hoffmann, 2003, p. 169).
In the problem of building towers of cubes and cuboids (see the section Combinatorial structuring strategies, conjectures and justifications), for example, nearly all five-tall towers could be found by the macro-strategy of complete inverses (see Figure 1).

In addition to English (1991, 1996) and Hoffmann (2003), the perspective of structuring is often crucial. In the example of tower building, from a horizontal perspective, further towers of the same height are found, whereas from a vertical perspective, the towers of a certain height are used to find the next higher towers (cf. Rütten and Weskamp, 2015). These perspectives imply special kinds of argumentation: horizontal structuring implies justification by cases whereas vertical structuring found a recursive justification (cf. Maher et al., 2011).

Therefore, the reconstruction of such structuring and its perspectives strategies is central since learners’ conjectures and their justifications are based on these. Sowder and Harel (1998) reconstructed classes of students’ justifications by interviews. Even though the authors do not relate their classification to combinatorial justifications explicitly, their proof schemes can be applied to students’ justifications as well to combinatorial SLE. According to the authors, justification means convincing oneself and convincing others (Sowder and Harel, 1998, p. 670, cf. also Harel, 2007). Therefore, they reconstructed three schemes of justification used by secondary students: externally based proof schemes, empirical proof schemes, and analytic schemes. Justifications of the first scheme class refer to some outside source (e.g. textbook). Justifications of the empirical proof scheme class are characterized either by the evidence from examples (inductive proof scheme) or by perceptions (perceptual proof scheme). The third proof scheme class is indicated by the essential ingredient of logical deduction (cf. Harel, 2007, p. 76). As a mathematical proof in the narrower sense, this class can hardly be expected from primary students (cf. Krummheuer and Fetzer, 2005, p. 27). At best, first buds of this could be reconstructed. However, a category occurring in primary students’ problem solving, that is not mentioned in the model of Sowder and Harel (1998), resp. Harel (2007), is the conjecture without justification, which was reconstructed by Lack (2009) by solving combinatorial problems (cf. also Hoffmann, 2003, p. 134).

All these theoretical backgrounds were leading the design and research of the following SLE.
Building towers of cubes and cuboids

In SLE ‘Fibonacci sequence’, three isomorphic combinatorial follow-up units have been developed which differ in the context. In the first SLE, pavements are paved by 1x2 oblong tiles (cf. Böttinger, 2006). In the second, towers of blue and yellow Lego bricks are built up without putting yellow bricks directly on each other (cf. Rütten et al., 2018). Finally, in the third SLE, different towers are built up by cubes (1 x 1 x 1) and special cuboids (1 x 1 x 2). In the following, the focus is only on this last SLE, because so far only in this one, learners’ conjecturing and its justification were examined in more detail.

![Figure 2: One to five-tall towers of cubes and cuboids](image)

In this SLE, the number of towers of a certain height is determined by the pattern of Fibonacci sequence. That is because a cube can be placed on each \((n-1)\)-tall tower, and a cuboid on each \((n-2)\)-tall towers to obtain the towers of the height \(n\) (see Figure 2).

In the SLE, small groups (6-10 students) work on this problem in pairs with concrete material and a worksheet. After building all towers of height 1, 2, 3, 4, and 5 and note them on the worksheet, the students are asked to make a conjecture about the number of six-tall towers and to justify it. Afterwards, there is an exchange of these conjectures and justifications in the whole group. In doing so, the justifications will be further developed.

**First results: Learners’ conjectures and schemes of justification**

After becoming familiar with Fibonacci sequence in the introductory unit by solving the rabbit problem adapted from Liber Abaci, the students reinvent the
sequence in a combinatorial context in the SLE ‘Building Towers of Cubes and Cuboids’. In all design experiments of this SLE, all children formulate conjectures. These are quite different and dependent on the numbers of smaller towers, which were found by them. For example, Laya and Lana, who found only 4 four-tall and 6 five-tall towers, supposed that there are only 6 six-tall towers (all examples taken from the project; translation CR/SW). However, most children supposed correctly that there are 13 six-tall towers and even try to justify their conjectures as it is asked on the worksheet. But some students – like Hayat – formulate only a conjecture without justification (see the section Combinatorial structuring strategies, conjectures and justifications):

Hayat: There are 13 possibilities.

However, these students are the exception. Most students try to provide a justification of their conjecture. These justifications can be categorized oriented towards classes of justification (see the section Combinatorial structuring strategies, conjectures and justifications).

Only a few students’ justifications refer to some outside source and can be called externally base justification. Toby supposed that there are 13 six-tall towers. With this number he associated Friday the 13th. This association he mentions as a justification.

Toby: Because I thought of Friday the 13th.

Most of the students’ justifications can be assigned to empirical proof schemes which “are marked by their reliance on either evidence from example […] or perceptions” (Harel, 2007, p. 67). Because of the data and in contrast to Sowder and Harel (1998) resp. Harel (2007), four sub-schemes can be distinguished: trial-and-error-based justification, figure-based justification, numeral justification, and unrelated figural and numeral justification.

Lack (2009, p. 209) indicates justifications based on the argument that nothing new can be found. This argument ultimately refers to a trial-and-error procedure that is performed until putatively all possibilities are found. However, the end of the procedure and thus the number of possibilities that have been found remain subjective and depending on the individual's ability to find no further possibilities. Such a so-called trial-and-error-based justification is formulated by Victoria.

Victoria: There are 13 possibilities because you must try until all possibilities are found.

This scheme of justification refers to the procedure only, but not to explicit listing of the corresponding possibilities. This is different with the figure-based justification. In doing so, students list all possibilities explicitly which they have found. Sometimes in these listings, micro-strategies can be reconstructed. So, Luke could have used inverses to find the fourth by the third tower and the sixth
by the fifth tower (see Figure 3, right). In Sedef’s drawing, it is hard to reconstruct any micro-strategy (see Figure 3, left).

![Figure 3: Six-tall towers of Sedef (left) and Luke (right)](image)

Both students did not find the full number of possible towers. Overall, the perspective in the figure-based justification seems rather horizontal. Relationships between five- and six-tall towers and thus a vertical perspective as a bud of a more sophisticated argumentation are not identifiable.

However, such a bud appears in the numeral justification. This justification refers to the numeral pattern which emerged in the structure of towers. It is close to what Sowder and Harel (1998) call example-based justification. The students focus solely on the numbers of towers without regard to the concrete towers. On this basis, they generate a counting strategy.

- **Leny:** There are 13 possibilities because it is like $1 + 2 + 3 + 5 + 8 = 13$.
- **Nicole:** Because of all results: $1 + 2 = 3 + 5 = 8$, so you have simply to calculate $5 + 8 = 13$.
- **Waieed:** Because it’s like $1 + 1 = 2 + 1 = 3, 2 + 3 = 5 + 3 = 8 + 5 = 13$.

Especially, when students want to express the relation of all numbers of towers with their calculations, mistakes in notation occur in which the meaning of the equal sign is disregarded. To show this relation, some students refer to the Fibonacci sequence which is like the discovered number pattern.

- **Elisa:** Because at height four there was an amount of 5 and at the fifth 8. $5 + 8 = 13$. That is Fibonacci-Sequence.

All kinds of numeral justification do not refer to the concrete construction of towers. Thus, they do not consider the risk of such reasoning which occurs by the possibility that the discovered pattern has no validity in general. This generality must be considered by more sophisticated justifications. Such justifications will be based on the interdependence of structuring and counting. This analytic justification (cf. Sowder and Harel, 1998) cannot be reconstructed from the data of primary students. However, there are students, who refer to both in their justifications, the concrete towers (structuring) as well as their number (counting).

- **Merle:** There are 13, [because at the others there are always so many towers as cubes] (deleted) you must add the number above plus the number above.
First, Merle’s justification refers to the construction of towers, where it cannot be reconstructed how Merle determines exactly the relationship between the towers and their number. However, she deletes this justification and describes how the number of six-tall towers can be found via the number pattern. Merle fails in relating the construction of towers to the identified number pattern. Her reasoning remains bounded by examples and can thus be called merely empirical. But this scheme can be named as an unrelated figural and numerical justification.

These different attempts of justifications on the worksheets were further developed or rejected in the subsequent exchange with the whole learning group. In doing so, the learners have increasingly taken a vertical perspective on their structuring. As part of the further development of a figural base justification, students refer to micro-strategies not only by using pattern from a horizontal perspective. They improve their justifications by using pattern from a vertical perspective to find further solutions.

Eda: Because here (points at tower ‘cube, cuboid’) looks like this (points at tower ‘cube, cube, cuboid’) and this one (points at tower ‘cuboid, cube’) looks like (points at tower ‘cuboid, cube, cube’) and this one (points at tower ‘cube, cube, cube’) looks like this one (points at tower ‘cube, cube, cube, cube’).

So, Eda structures the set of towers by comparing single towers from a vertical perspective. However, the vertical structuring made in this way cannot be related to the occurring number pattern yet. This bud of more sophisticated justifications thus remains at the level of a new perspective on structuring.

**Conclusions and perspectives**

For the moment, schemes of analytic justification could not be reconstructed in primary students’ argumentations in solving the problem of building towers by cubes and cuboids. But the students use different kinds of less sophisticated justifications, whereby some relate to their structuring strategies. These structuring strategies are based on a horizontal perspective (e.g. inverses) on the one hand, and a vertical perspective (e.g. structuring towers of different height which only consist of cubes) on the other hand. Toward an analytic justification, teacher’s intervention for improving justifications could be helpful: “How can six-tall towers build up by five- and by four-tall towers?” In addition, a fictional student’s solution may be suitable to foster students’ justifications in the sense of the analytic proof scheme by representing all towers of height 1, 2, 3, 4, and 5 in a vertical structure like in Figure 2. This intervention should be explored in the further design cycle. Perspectively, the local theories from this design research process should be compared with findings from the other two combinatorial follow-up units and developed further mutually.
References


MATHEMATICAL MODELLING IN PRIMARY EDUCATION. 
OPINIONS OF SLOVENIAN AND CROATIAN TEACHERS ON 
TEACHING MATHEMATICAL MODELLING IN EARLY PRIMARY 
EDUCATION

Mateja Sabo Junger and Alenka Lipovec

Abstract

As the world as well as the STEM area, progresses, we see the need for a growing number of young people who are good at connecting, constructing and modelling everyday problems. This is just one of the reasons why we need mathematical modelling, and the need to start practising it in elementary schools as much as possible. We will describe what mathematical modelling is, show what are the advantages, disadvantages, how to teach mathematical modelling in primary school and what benefits students will have from it. We also present the results of the research, in which we studied opinions of Slovenian and Croatian teachers regarding mathematical modelling in early primary education. The survey was conducted in Slovenia and Croatia on a sample of 626 teachers of the first five/four years of elementary education. We asked questions about the general meaning of mathematical modelling, gave them the definition of the same, gave a sample of task related to mathematical modelling, set some questions related to the task. Furthermore, we set up claims about mathematical modelling with which they could either agree or not, and finally a few questions about their competency and obstacles in learning mathematical modelling.

Keywords: mathematics, modelling, teacher education, primary education

Theoretical framework

There are many definitions of mathematical modelling, and in our research we have decided for the following: Mathematical modelling is an iterative process that involves open, realistic, practical problems that pupils explain by using mathematics using assumptions, approximations, and multiple representations.

Figure 1: Cycle of mathematical modelling

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They can also use other sources of knowledge, not just mathematical (Stohlmann and Albarracin, 2016). To better illustrate the process of mathematical modelling, we can use the cycle of mathematical modelling from Blum and Leiss (2007) (see Figure 1).

In the modelling circle, we begin with the problem that is to be solved using math tools. In the first stage the problem is described by relevant non-mathematical terms. During this phase it is necessary to choose (simplify) assumptions. The outcome of the first phase is the conceptual model. This conceptual model is then translated into a mathematical model that can be mathematically analyzed. Furthermore, the mathematical solution is translated back into the language of the initial problem, what is called interpretation. Finally, we confirm the solution. If necessary, we will again begin the modelling circle by adjusting one or more steps (Spandaw and Zwaneveld, 2009).

Traditional mathematical modelling at elementary school focused on arithmetic problems with words (tasks with words) in which concrete materials were presented that would then model the more abstract operating rules (English). Solving such tasks requires a link between problem structure and symbolic mathematics. For example: *Suzana saved $ 12. Ljiljana saved three times more than Suzana. How much did Ljiljana save?* The solution is modeled: $12 \times 3 = 36$. Often solving such problems is not a modelling task for pupils, but relies on keywords or phrases in the problem, such as times, more, less, and so on (English). Furthermore, there is often only one way of interpreting the problem, so children are included in limited mathematical thinking. Although we do not dispute the importance of this type of tasks, they do not address enough mathematical knowledge, processes, fluency, and social skills that our children in the 21st century need (English, 2003).

In general, some of the goals of mathematical modelling are:

- helping students to better understand the world around them,
- supporting mathematics learning (motivation, concept creation, understanding, retention),
- contributing, to the development of various mathematical competences and appropriate attitudes,
- contributing, to the appropriate picture of mathematics. With modelling, math becomes more significant for students (Blum and Borromeo Ferri, 2009).

Modelling is mostly related to out-of-mathematical contexts. Consequently, a teacher can easily find himself in an awkward situation as he can not be an expert on all possible fields of modelling, such as natural sciences, computing, economics, art, sports and so on. The same goes for the students. It is therefore important to choose the context of the model for which both the teacher and the students have sufficient knowledge. The teacher must encourage students to use
that knowledge, which does not have to be closely related to mathematical knowledge. It also needs the teacher to be aware that a certain problem can lead to different models (Spandaw and Zwaneveld, 2009). One of the major obstacles in learning mathematical modelling is the evaluation. Modelling goals cannot be assessed as objective as it is usual in the teaching of mathematics. Teachers who take seriously into account non-mathematical aspects of modelling must face a lack of objectivity. In order to reduce the subjectivity, it is proposed to use the modelling circle as the basis for the creation of criteria: conceptualization (initial problem analysis, data, variables, links, simplifications, the goal of modelling), mathematical analysis (completeness, accuracy), interpretation, confirmation, conclusions, adaptation. Commonly written tests are not well suited to evaluating higher skills such as modelling. It would be better to use other alternatives, such as group work, homework or oral exams (Spandaw and Zwaneveld, 2009). The activities of mathematical modelling for children should be based on their existing understanding and involve them in thought-provoking multiple problems involving the participation of smaller groups. Such activities should be set up in authentic contexts that allow multiple different interpretations and approaches. In such activities, children deal with important mathematical processes such as description, analysis, coordination, explanation, design, and critical thinking of things, relationships, patterns, and rules (English, 2003).

**Research problem**

The aim of the research is to examine Slovenian and Croatian teachers’ opinions on mathematical modelling in early primary education. The questions were set in Croatian for teachers from Croatia and Slovenian language for teachers from Slovenia. The questions we asked the teachers' answers or opinions were:

1. Have they met at all with the concept of mathematical modelling and whether they do tasks/activities involving mathematical modelling in their work with students?
2. We provided an example of a task that includes mathematical modelling and asked them how they would evaluate it, and whether they would include such a task in their math lessons.
3. Questions related to the very definition of mathematical modelling (which we wrote in the survey) and the connection between incorporating such activities in the teaching of mathematics with possible obstacles that might arise.
4. Do they think they are sufficiently educated to teach mathematical modelling, and if not why it is so?

**Methodology**

We have used the methods of quantitative empirical pedagogical research. The survey was carried out on the basis of completed questionnaires on a convenience
sample of 626 teachers from Slovenia and Croatia. The survey was conducted in February and March of 2019.

**Instrument**

A questionnaire was designed with several types of questions: a) questions about teachers' basic data (gender, years of work experience in classroom), b) general questions about mathematical modelling, c) questions about a given specific mathematical modelling task d) claims regarding mathematical modelling given with a five-degree scale of agreement/disagreement e) questions about teachers’ opinions about introducing mathematical modelling into classroom.

**Sample**

Participants were teachers from Slovenia and Croatia (N = 626; 3% male, 97% female). The structure of the sample according to the country is presented in Table 1.

<table>
<thead>
<tr>
<th>Country</th>
<th>f</th>
<th>f %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slovenia</td>
<td>200</td>
<td>32%</td>
</tr>
<tr>
<td>Croatia</td>
<td>426</td>
<td>68%</td>
</tr>
</tbody>
</table>

Table 1: Sample structure

**Results**

In the continuation of the article, we will use the abbreviation MM – mathematical modelling. First, we present the results regarding whether they ever met with the term MM in Table 2.

<table>
<thead>
<tr>
<th>Have you met with the concept of mathematical modelling?</th>
<th>f</th>
<th>f %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slovenia</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>17</td>
<td>9</td>
</tr>
<tr>
<td>No</td>
<td>166</td>
<td>83</td>
</tr>
<tr>
<td>In passing</td>
<td>16</td>
<td>8</td>
</tr>
<tr>
<td>Croatia</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>33</td>
<td>8</td>
</tr>
<tr>
<td>No</td>
<td>345</td>
<td>82</td>
</tr>
<tr>
<td>In passing</td>
<td>42</td>
<td>10</td>
</tr>
<tr>
<td>Combined</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>50</td>
<td>8</td>
</tr>
<tr>
<td>No</td>
<td>511</td>
<td>83</td>
</tr>
<tr>
<td>In passing</td>
<td>58</td>
<td>9</td>
</tr>
<tr>
<td>∑</td>
<td>619</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 2: General question about mathematical modelling

From the results, we can see that teachers from Slovenia and Croatia have not met with the concept of mathematical modelling (83%). Second, we asked them if they do tasks or activities that include MM in their math classes. We present the results in Table 3.
Do you use tasks/activities in your math class that include mathematical modelling?

<table>
<thead>
<tr>
<th></th>
<th>f</th>
<th>f %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slovenia</td>
<td>Yes</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>133</td>
</tr>
<tr>
<td></td>
<td>Sometimes</td>
<td>16</td>
</tr>
<tr>
<td>Croatia</td>
<td>Yes</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>264</td>
</tr>
<tr>
<td></td>
<td>Sometimes</td>
<td>44</td>
</tr>
<tr>
<td>Combined</td>
<td>Yes</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>No</td>
<td>397</td>
</tr>
<tr>
<td></td>
<td>Sometimes</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td>∑</td>
<td>495</td>
</tr>
</tbody>
</table>

Table 3: General question about mathematical modelling

From Table 3 we can see that 80% do not use any tasks or activities that include MM, which could be expected given to the results of the previous question (Table 2). Furthermore, in the survey, we have written the above definition of MM and the following task: Farmer Jaka is trying to decide which light conditions are best for growing beans. To help Farmer Jaka make his decision, he is growing bean plants using two different light conditions. The two light conditions are:

Growing beans in the full sun with no shade at all, and growing beans in the shade (Figure 2).

<table>
<thead>
<tr>
<th>Butter Bean Plants</th>
<th>Sunlight</th>
<th>Week 6</th>
<th>Week 8</th>
<th>Week 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td></td>
<td>9 kg</td>
<td>12 kg</td>
<td>13 kg</td>
</tr>
<tr>
<td>Row 2</td>
<td></td>
<td>8 kg</td>
<td>11 kg</td>
<td>14 kg</td>
</tr>
<tr>
<td>Row 3</td>
<td></td>
<td>9 kg</td>
<td>14 kg</td>
<td>18 kg</td>
</tr>
<tr>
<td>Row 4</td>
<td></td>
<td>10 kg</td>
<td>11 kg</td>
<td>17 kg</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Butter Bean Plants</th>
<th>Shade</th>
<th>Week 6</th>
<th>Week 8</th>
<th>Week 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row 1</td>
<td></td>
<td>5 kg</td>
<td>9 kg</td>
<td>15 kg</td>
</tr>
<tr>
<td>Row 2</td>
<td></td>
<td>5 kg</td>
<td>8 kg</td>
<td>14 kg</td>
</tr>
<tr>
<td>Row 3</td>
<td></td>
<td>6 kg</td>
<td>9 kg</td>
<td>12 kg</td>
</tr>
<tr>
<td>Row 4</td>
<td></td>
<td>6 kg</td>
<td>10 kg</td>
<td>13 kg</td>
</tr>
</tbody>
</table>

Figure 2: Table with data on the growth of beans in the sun and in the shade

Using the data above, determine which of the light conditions is suited to growing beans to produce the greatest crop. In a letter to Farmer Jaka, outline your recommendation of the light condition and explain how you arrived at this decision. Second, predict the weight of butter beans produced on week 12 for each type of light. Explain how you made your prediction so that Farmer Jaka can use it for other similar situations (English, 2003).

After a given assignment, we asked the teachers what they think is the best way to evaluate the task. The results are shown in Table 4.
Which is the best way to evaluate the task?

<table>
<thead>
<tr>
<th></th>
<th>f</th>
<th>f %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oral exam</td>
<td>92</td>
<td>32</td>
</tr>
<tr>
<td>Written exam</td>
<td>27</td>
<td>9</td>
</tr>
<tr>
<td>Group work</td>
<td>122</td>
<td>42</td>
</tr>
<tr>
<td>Homework assignment</td>
<td>13</td>
<td>4</td>
</tr>
<tr>
<td>This task can’t be evaluated</td>
<td>17</td>
<td>6</td>
</tr>
<tr>
<td>Other</td>
<td>20</td>
<td>7</td>
</tr>
<tr>
<td><strong>∑</strong></td>
<td>291</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 4: Question about a given specific mathematical modelling task

The results show that the majority of teachers agree that the written exams are not the best option when it comes to evaluating tasks that include MM, but that oral examination or group work would be a better option. If they answered other, some of the propositions were: all in combination, presentation, self-evaluation, research task.

Furthermore, we presented the teachers eight claims related to MM, which they could answer with five-degree scale (strongly agree, partially agree, neither agree nor disagree, partially disagree, and completely disagree). The two out of eight statements and results are shown in Table 5.

<table>
<thead>
<tr>
<th>Combined (SLO + CRO)</th>
<th>f</th>
<th>f %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical modelling is exact, formal process, or a collection of formulas and rules that have to be applied.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>completely disagree</td>
<td>64</td>
<td>23</td>
</tr>
<tr>
<td>partially disagree</td>
<td>39</td>
<td>14</td>
</tr>
<tr>
<td>neither agree nor disagree</td>
<td>80</td>
<td>29</td>
</tr>
<tr>
<td>partially agree</td>
<td>75</td>
<td>28</td>
</tr>
<tr>
<td>completely agree</td>
<td>17</td>
<td>6</td>
</tr>
<tr>
<td><strong>∑</strong></td>
<td>275</td>
<td>100</td>
</tr>
</tbody>
</table>

Although mathematical modelling activities improve students' ability to solve problems, I find that there are too many obstacles to incorporate such activities into my math classes.

<table>
<thead>
<tr>
<th></th>
<th>f</th>
<th>f %</th>
</tr>
</thead>
<tbody>
<tr>
<td>completely disagree</td>
<td>55</td>
<td>20</td>
</tr>
<tr>
<td>partially disagree</td>
<td>54</td>
<td>20</td>
</tr>
<tr>
<td>neither agree nor disagree</td>
<td>62</td>
<td>22</td>
</tr>
<tr>
<td>partially agree</td>
<td>94</td>
<td>34</td>
</tr>
<tr>
<td>completely agree</td>
<td>12</td>
<td>4</td>
</tr>
<tr>
<td><strong>∑</strong></td>
<td>277</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 5: Claims regarding mathematical modelling
From the results we can see that although we have already defined the MM before these claims, teachers in large percentages even 34% fully or partially agree with the statement that MM is exact, formal process, or a collection of formulas and rules that have to be applied. From the results of the second claim, we also notice that a large percentage of teachers (38%) think that there are too many obstacles to implement activities involving MM, although they know that MM improves many student capabilities, such as solving problems.

The last set of questions related to the teachers' thoughts on whether they think they were educated enough to teach MM in elementary math classes, and if they thought they were not, what was the reason for that. Results are shown in Table 6.

<table>
<thead>
<tr>
<th></th>
<th>f</th>
<th>f %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Combined</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yes</td>
<td>67</td>
<td>24</td>
</tr>
<tr>
<td>No</td>
<td>208</td>
<td>76</td>
</tr>
<tr>
<td>Σ</td>
<td>275</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 6: Question about teachers’ opinions about introducing mathematical modelling into classroom

Most teachers feel that they are not educated enough to teach MM (76%). Some of the reasons they mentioned were: insufficient workshops/education/professional training on this subject, lack of examples and materials, have never met this term, too little experience in working with MM and so on.

**Discussion**

Application and modelling are important, and learning application and modelling are demanding. This implies that there must be particularly great efforts to make the application and modelling accessible to students. There are such efforts in many countries around the world. However, in the daily teaching of mathematics in most countries, there is still relatively little modelling (Blum, 2015). We have seen that even 83% of primary school teachers from Croatia and Slovenia have never met with the concept of MM, while 9% of them met in passing. The results have shown that most teachers do not use examples/tasks/activities that include MM in their math classes. Modelling almost always contains non-mathematical contexts. Consequently, a teacher can easily find himself in an embarrassing situation as he can not be an expert on all possible fields of modelling, such as natural sciences, computing, economics, art, sports and so on. The same goes for the students. It is therefore important to choose the context of the model for which both the teacher and the students have sufficient knowledge. The teacher must encourage students to use that knowledge, which does not have to be closely related to mathematical knowledge. It also needs the teacher to be aware that a certain problem can lead to different models (Spandaw and Zwanenveld, 2009). After describing the definition of mathematical modelling in the second part of
the survey, and the example of a task involving MM, the teachers agreed that such task would be difficult to evaluate using conventionally written exams, but group work or verbal examination would be much more appropriate. One of the major obstacles to learning mathematical modelling is evaluation. Modeling goals can not be assessed as objectively as it is usual in the teaching of mathematics. Commonly written tests are not well suited to evaluating higher skills such as modelling. It would be better to use other alternatives, such as group work, homework or oral exams (Spandaw and Zwaneveld, 2009). In this paper, we have further outlined two statements that teachers could agree with or not. In the first claim that mathematical modelling is an exact, formal process or collection of formulas and rules to be applied, 37% of teachers did not agree or partially disagreed with the statement, while 34% of the teachers partially or completely agree with the statement, although earlier in the poll was given definition of MM, from which it was clearly seen that MM was not that. With the second statement in this article that says, that although mathematical modelling activities improve students ability to solve problems, I find that there are too many obstacles to incorporate such activities into teaching math, 40% of teachers have either partially or completely disagreed, while 38% partially or completely agreed. The reason for this may be that the teachers were poorly educated on MM, as they themselves expressed in the survey. Mathematical modelling can be effectively learned by students of different ages, but it requires effort and investment in the sense of careful and focused teaching design, learning environments, activities and time to develop such activities and tasks. Modelling skills can not be developed by teaching that is focused on typical examples and tasks in the hope that this will result in knowledge of modeling real-world situations and tasks (Niss, 2012). In the last set of questions, the results showed that teachers are aware of the fact that they are not educated enough to teach mathematical modelling (76%) and that they need additional workshops/training. In modelling, students will face the problems that matter to them and the society in which they live. They will have to decide which information is relevant, make approximations and wisely use appropriate mathematical tools. As teams, students will persevere through challenges, and surprise us with the ways they can use mathematics to improve the world we live in. (English, 2003). In this paper, we have only presented a part of the survey, and the questions from the survey questionnaire. Undoubtedly we know that mathematical modelling is particularly difficult and complicated for teachers, but many studies show and point to its benefits and well-being for students. Certainly, it is obvious that a lot more education for teachers is needed to enable them to teach mathematical modelling.

References


THE INFLUENCE OF DIAGRAMMATIC REASONING ON A TEACHER’S MATHEMATICAL EXPERIENCE

Adalira Sáenz-Ludlow and Alexandra Jiménez Jiménez

Abstract

From the Peircean perspective of diagrammatic reasoning, the paper analyzes a teacher’s reflection on a task, of her choosing, involving a square array of dots. The teacher goes beyond the proposed partition of the array and proposes two new partitions to arrive at new ways of conceptualizing square numbers. In addition, she proposes each task as a sequence of square arrays of dots to set in motion inductive reasoning in order to arrive to generalizations. By doing so, the teacher displays the influence that the notion of diagrammatic reasoning had on her mathematical way of thinking.

Keywords: square numbers, diagrammatic reasoning, teacher’s mathematical experience

Theoretical rationale

Peirce’s triadic SIGN. The Peircean SIGN is constituted by the sign-vehicle, the interpretant, and the Object along with three dyadic relations among them. These relations are between (1) the sign-vehicle and the Object it plays to represent; (2) the sign-vehicle and the interpretant it determines in the mind of the Interpreter; and (3) the Object and the interpretants generated by the Interpreter. It is important to note that the interpretant is not the Interpreter. The interpretant is the effect of the sign-vehicle on the mind of the Interpreter who is the agent that takes part in and exerts control over his/her own process of interpretation. The interpretant of a sign-vehicle, in the mind of the Interpreter, depends on what the Interpreter

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‘makes of it’ and evolves continuously. This transformation of interpretants into more developed interpretants (an infinite semiosis) generate dynamic objects that have a tendency toward the Object of the SIGN. The Object of a sign-vehicle is not transparent to the Interpreter, all at once, because the Object possesses many aspects and one sign-vehicle can only represent some of its aspects and not all the aspects at the same time. This clearly indicates the need to represent any Object/Concept by means of more than one sign-vehicle.

Peirce (1992) considers that the relation between the Object and the sign-vehicle could be iconic, indexical, or symbolic according to both how it represents the Object and how it is interpreted by Interpreter. Fisch (1986) explains that this relation between the sign-vehicle and its Object is dynamic and evolves in relation to the Interpreter and the sophistication of his/her collateral and prior knowledge. Thus, we have to ask for whom the sign-vehicle is iconic, indexical, or symbolic.

A diagram is defined (Peirce, 1992) as an icon of possible relations. He argues that the diagram has structural similarities with the abstract and hidden structure of its Object. This similarity warrants the purposeful observation by means of inferential reasoning (inductive, abductive, and deductive) of the structural relations among the parts of the physical diagram (the phenomenon or the Object-as-it-is-perceived) and the parts of the Object (the noumenon or the Object-as-it-is).

The Interpreter may see a diagram merely as a token (a pure icon with very limited meaning), or as a schema (an icon with indexical traits of some type of structure), or as a symbol (an icon with iconic-indexical-symbolic traits emerging from the intellectual relations of the Interpreter). Then what type of sign-vehicle is a diagram? It could be one of the three kinds (icon, index, or symbol); it depends on what the Interpreter ‘makes of it.’ This means that it depends both on the prior knowledge of the Interpreter and on the collateral knowledge she or he is able to draw into the situation at a particular point in time. This is to say that the Interpreter, in the process of observation and interpretation, simultaneously plays the perceptual elements in thought and the thought elements in perception.

**Diagrammatic reasoning.** Stjernfelt (2007) captures the essence of the Peircean process of diagrammatic reasoning as a process which is rooted in perceptual and mental activity to produce chains of interpretants. Diagrammatic reasoning synthesizes a manifold of relations that integrates the construction/observation of a diagram, the observation of structural relations among its parts, and the perceptual manipulation and thought-experimentation to infer new possible relations conducive to the attainment of the conceptualization of the Object of the sign-vehicle (the Object-as-it-is). He also describes this process in terms of the transformation of diagrams co-emerging with the formation of evolving interpretants. This is to say that, during the process of interpretation, the Interpreter mentally refurbishes the given or initially constructed diagram.
(transformand diagram) into more meaningful diagrams (transformate diagrams) to infer the hidden abstract Object that the diagram purport to represent.

**Diagrammatic reasoning and visualization.** The aim of diagrammatic reasoning is the construction of a mathematical argument that warrants the abstract structure of the mathematical Object. The formation of the argument is, for Peirce, the formation of a logical rule that has coherence and completeness. This argument is constituted not only by the construction of isolated inferences but also by the logical and cohesive concatenation of them. Each inference is the result of evolving chains of related interpretants to form a logical assertion. The argument is the concatenation of logical assertions that lend themselves to form a coherent chain of mathematical inferences.

Bishop (1989) sheds light on two processes: (1) integrating figural information or the ability to translate a particular situation and to present it in some visual form and (2) visual processing or the ability to translate abstract relations and nonfigural information into visual terms. These two processes are encapsulated by Arheim (1969) in his seminal book *Visual Thinking*, in which he rephrases Kant to give emphasis to visualization: “vision without abstraction is blind and abstraction without vision is empty” (p. 188). This is to say that both visual perception and conceptualization go hand-in-hand and, one without the other appear to be almost impossible. Thus, visualization mediates the emergence of diagrammatic reasoning to arrive to generalization (Sáenz-Ludlow, 2018).

**Methodology**

**The Subject.** The teacher here described is one of six doctoral students who were participating in a graduate seminar on diagrammatic reasoning. The participants were all teachers (school teachers or university instructors) with three to eight years of teaching experience. The data here analyzed comes from a teacher with six years of experience teaching school children as well as practicing elementary school teachers.

**The Method.** The seminar lasted 20 hours in which different articles on Peircean semiotics and diagrammatic reasoning were studied and reflected upon. Other articles focused on diagrammatic reasoning as expanded by other thinkers using the Peircean theoretical framework (Stjernfelt, 2007). The participants were required to write weekly reflections on their understanding of diagrammatic reasoning and how this notion could be incorporated in their teaching practice. Each participant could use their own tasks to illustrate the possibility to change or modify their own teaching practice. These papers were used by the seminar’s instructor to clarify and/or extend the understanding of the participants on diagrammatic reasoning. The last paper included the choosing of one task used before in their teaching and thinking it again through the notion of diagrammatic reasoning and how this could lead to a richer mathematical experience for the participant himself/herself and for his/her students. After submission of the final
paper, the researcher interviewed three times the participating teacher. Each interview lasted hour-and-a-half. The main purpose of the interviews was to assess whether or not the teachers’ interpretation of the elements of diagrammatic reasoning could influence her mathematical thinking and her practice.

**The Task.** The teacher chose a diagram from Nelsen (1993) book “Proof Without Words: Exercises in Visual Thinking” (vol. 1). The implicit purpose of the diagram was to arrive to the generalization that “all square numbers can be written as a sum of consecutive odd numbers. (See Figure 1.)

The author (or proposer) of the diagram skillfully visualizes an abstract relation about square numbers into a visual square array of dots and partition it into L’s, each of which is formed by an odd number of dots. This is an instance of what Bishop (1989) calls visual processing or the ability to translate abstract relations and no figural information into a visual format.

![Transformand diagram/ transformand task](image.png)

Figure 1: Transformand diagram/ transformand task

The teacher, instead, used the reverse process and generated particular sequences of arrays of dots to arrive to the generalization implicitly requested or other generalizations anticipated by her. This is an illustration of what Bishop (1989) calls integrating figural information, or the ability to relate to a particular situation and to present it in some visual form. The teacher transforms the initial diagram (i.e., transformand diagram) into a sequence of consecutive arrays of dots to facilitate inductive thinking and the emergence of the generalization about square numbers. The teacher also generates three more sequences, each of which visualizes the square array of dots partitioned in different ways to arrive at different generalizations about square numbers. That is, the teacher modified the problem to pose different conceptualizations of square numbers.

**Analysis**

The analysis incorporates the explanations of the teacher when she was interviewed. The teacher’s reflections and transformations of the initial diagram emerged both from her own interpretants and from her visualization of other partitions of the square array of dots. These new partitions mediated the construction of other tasks in the form of particular sequences.
The teacher was well aware that a diagram synthesizes a manifold of relations that could be explored by means of thought-experiments. By visualizing it (perceptually and conceptually) in more than one way, she anticipated that different partitions of the square array of dots could imply different generalizations about square numbers. The transformation of the initial diagram co-emerged with her construction of new *interpretants*. This is to say that, during her process of interpretation, she mentally refurbished her *interpretants* of the initial diagram (transformand diagram) to generate a sequence of diagrams (transformate diagram) from which different generalizations about square numbers could emerge. Another extension of the given square array of dots was to see the product \( n^2 \) as the result of *n* iteration of the *n* dots in one row.

**Square numbers as the sum of consecutive odd numbers**

In Figure 2, the teacher transforms the initial diagram into a sequence of arrays to facilitate inductive thinking. The mathematical description of the first few squares amounts to the actual counting of the dots on each of the L’s that constitute each square. However, when we have a square with \( n^2 \) number of dots: What is the number of dots in the last L or what is the form of the odd number of dots in the \( n^{th} \) L?

![Figure 2: Transformate diagram: A sequence of square arrays of dots](image)

Each side of the L is formed by *n* dots, but there is one dot, the one on the corner, that is counted twice. So, the last L could be seen as \( 2n - 1 \) which is equivalent to \( 2(n - 1) + 1 \). Using the visual pattern, the teacher constructs Table 1 and verifies that the last odd number on each row is of the form \( 2n - 1 \). In the table, which is in itself a diagram, the pattern reveals itself: the *square* of any *natural number* *n* is the sum of the first *n* *consecutive odd numbers* starting with 1. The teacher constructs here a sequence of logical interpretants, that helps her to construct the table, one step at a time, and to establish each particular case, before she induces the \( n^{th} \) case with \( n^2 \) dots in the array.
A recursive generalization

The teacher also visualizes a partition of each square using the prior square in the sequence and adding an appropriate L on the upper right side and with a convenient number of dots to form the following square (see Figure 3). So, going backwards in the sequence, one could implicitly conclude that the square of any natural number \( n \) is the sum of the first \( n \) consecutive odd numbers starting at 1. In other words, the square with \( n^2 \) dots is constructed out of \( n \) increasing L’s. So, she recursively arrives at the same generalization as before although through different means that were the result of her visual and logical interpretants.

Using the visual pattern, the teacher constructs Table 2 (an organizational diagram) to record her numerical observations in an organized manner. Row by row, the pattern reveals itself. Reading from the table it can be expressed, in natural language, that any square number can be constructed using the prior square number and the L constituted by the appropriate number of dots to enlarge the square one more time. Without the teacher’s constructions of a chain of logical interpretants, the construction of the table, one step, at a time would have been impossible.
Table 2: Square arrays formed by a prior square array and a new L

Square numbers as the sum of triangular numbers

Once again, the teacher visually foresees a partition of each square in the sequence using the diagonal of each square and the two triangles it determines. She immediately sees each triangle constituted by a triangular number of dots (i.e., a triangular array of dots, formed by decreasing by one the number of dots on the diagonal until one arrives at 1 dot). She then transforms the task into the sequence of squares in Figure 4 to favor the emergence of inductive thinking.

She organizes her visual observations on each square in Table 3. Her records reveal the pattern she foresaw. The square in the nth position (i.e., the square with $n^2$ dots) is the sum of the n dots in its diagonal plus twice the ($n - 1$)th triangular number. The table is a product of the teacher’s construction of a new chain of logical and visual interpretants.

<table>
<thead>
<tr>
<th>Term</th>
<th>Total number of dots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1$^{st}$</td>
<td>diagonal dots $= 1 + 2(0) = 1 = 1^2$</td>
</tr>
<tr>
<td>2$^{nd}$</td>
<td>diagonal dots $+ 2$(first triangular number) $= 2 + 2(1) = 2 + 2 = 4 = 2^2$</td>
</tr>
<tr>
<td>3$^{rd}$</td>
<td>diagonal dots $+ 2$(second triangular number) $= 3 + 2(3) = 3 + 6 = 9 = 3^2$</td>
</tr>
<tr>
<td>4$^{th}$</td>
<td>diagonal dots $+ 2$(third triangular number) $= 4 + 2(6) = 4 + 12 = 16 = 4^2$</td>
</tr>
<tr>
<td>5$^{th}$</td>
<td>diagonal dots $+ 2$(fourth triangular number) $= 5 + 2(10) = 5 + 20 = 25 = 5^2$</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
</tr>
</tbody>
</table>
Table 3: Mathematical description of squares formed by the number of dots on the diagonal and two triangular numbers

The teacher explained the last line on the table as follows:

\[
\text{Number of dots on the diagonal of the } n^{th} \text{ square} = n \\
\text{triangular number } (n - 1)th = (n - 1) + (n - 2) + \cdots + 3 + 2 + 1 \\
= \frac{(n - 1)n}{2} \\
2 \text{ (triangular number} (n - 1)th) = 2 \frac{(n - 1)n}{2} = n(n - 1)
\]

Square numbers: Multiplication as repetitive addition

The teacher pursues the opportunity to make the connection between iterative addition and a special case of multiplication of natural numbers. Using the same sequence of squares and partitioning each square of the sequence into rows. Each row in the sequence is constituted by one, two, three, …, or \( n \) dots (i.e., units of cardinality one, two, three, …, or \( n \)) and in any square there are as many rows as there are dots in each row. (See Figure 5.) The question now is how to use these rows (i.e., units) to find the number of dots in each square.

![Figure 5: Transformate task: Using the number of dots on each row to create a visual linkage between multiplication as the iteration of units](image)

She explains her strategy following different lines of reasoning. One is to take each row (i.e., the unit of cardinality \( n \)) and add it to itself as many times as the number of rows in the \( n^{th} \) square.

\[
n + n + \cdots + n = \text{Unit of cardinality } n \text{ added } n \text{ times} = n(n) = n^2
\]
The other is to multiply the number of dots in each row (**a unit of cardinality** \(n\)) by the number of rows (also **a unit of cardinality** \(n\)) to determine the number of dots in the \(n\)th square.

**Unit of card. \(n\) (row) \textbf{times} Unit of card. \(n\) (number of rows) = \(n(n)\)

\[ = \text{unit of card. multiplied by itself} = n^2 \]

Then she organizes her observations in Table 4, one square at a time, to arrive at the generalization. Here again a new chain of logical and visual interpretants sets in motion the construction of the table.

<table>
<thead>
<tr>
<th>Term</th>
<th>Total number of dots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>\textit{add one time} the unit of cardinality (1 = 1 = 1^2) \textit{multiply by itself} the Unit of card. (1 = 1 \text{ time} 1 \times 1 = 1 = 1^2)</td>
</tr>
<tr>
<td>2nd</td>
<td>\textit{add two times} the Unit of card. (2 = 2 \text{ plus} 2 = 2 + 2 = 4 = 2^2) \textit{multiply by itself} the Unit of card. (2 = 2 \text{ times} 2 \times 2 = 4 = 2^2)</td>
</tr>
<tr>
<td>3rd</td>
<td>\textit{add three times} the Unit of card. (3 = 3 \text{ plus} 3 \text{ plus} 3 = 3 + 3 + 3 = 9 = 3^2) \textit{multiply by itself} the Unit of card. (3 = 3 \text{ times} 3 \times 3 = 9 = 3^2)</td>
</tr>
<tr>
<td>4th</td>
<td>\textit{add four times} the Unit of card. (4 = 4 \text{ plus} 4 \text{ plus} 4 \text{ plus} 4 = 4 + 4 + 4 + 4 = 16 = 4^2) \textit{multiply by itself} the Unit of card. (4 = 4 \text{ times} 4 \times 4 = 16 = 4^2)</td>
</tr>
<tr>
<td>5th</td>
<td>\textit{add five times} the Unit of card. (5 = 5 \text{ plus} 5 \text{ plus} 5 \text{ plus} 5 \text{ plus} 5 = 5 + 5 + 5 + 5 = 25 = 5^2) \textit{multiply by itself} the Unit of card. (5 = 5 \text{ times} 5 \times 5 = 25 = 5^2)</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(n)th</td>
<td>\textit{add (n) times} \textit{the Unit of card.} (n = \underbrace{n \text{ plus} n \text{ plus} n \text{ plus} n \ldots \text{ plus} n}<em>{n \text{ times}}) \textit{multiply by itself} \textit{the Unit of card.} (n = \underbrace{n \text{ times} n \times n = n^2}</em>{n \text{ times}})</td>
</tr>
</tbody>
</table>

\[ = \text{Unit of card. } n \text{ \textbf{times} unit of card. } n = n \times n = n^2 \]

**Table 4:** Mathematical description of squares formed by rows of dots

**Conclusion**

The above analysis indicates the way this teacher experiences anew a mathematical task she has solved before. Before being introduced to diagrammatic reasoning, she solved this task by visual perception to factually “see” that, in effect, a square could be constructed by accumulating consecutive L-shapes of dots and that each L necessarily had to have an odd number of dots. The fact that the particular case presented in the initial diagram was hinting at a general statement was good enough for her, personally, but not enough to convince other learners (children and teachers) and to guide their mathematical experience.
Knowing how-to-act-in-the-moment in the classroom requires time for conceptual reflection about the merits of tasks and possible modifications to guide the mathematical experience of learners (Mason, 2002). The more relations the teacher constructs, the greater the potential to help students build up their mathematical experience and mathematical connections.

The teacher noticed that the format of the task gave no opportunity for inductive thinking, reason for which she developed sequences of diagrams, so that students could make the transition from particular cases to general cases. The teacher’s transformation of the task emerged from her own interpretants, mediated by visualization and diagrammatic reasoning, which in turn, allowed her to deepen her mathematical experience and to increase the potential for a richer mathematical experience for her students.

Acknowledgements: This research was funded by the Fulbright Scholar Program, the European Commission and the ACACIA Project (561754-EPP-1-2015-1-CO-EPKA2-CBHE-JP).

References

**PREPARING FUTURE TEACHERS FOR FORMATIVE ASSESSMENT: THE CASE OF CONCEPT CARTOONS**

*Libuše Samková* ✉

Abstract

In this contribution I will address the issues related to the use of formative assessment in inquiry based mathematics education. I will show how an educational tool called
Concept Cartoons may be used in university training of future teachers to prepare them for future implementation of formative assessment into their school practice. The text introduces the background of inquiry based mathematics education and Concept Cartoons that relates to formative assessment, and describes a small qualitative empirical study conducted with future primary school teachers. The study focuses on aspects related to on-the-fly formative assessment that appeared in future teachers’ written responses to Concept Cartoons and in a subsequent discussion.

**Keywords**: Concept Cartoons, formative assessment, future teacher education, inquiry based mathematics education

**Introduction**
This paper is devoted to relation among formative assessment, inquiry based mathematics education, university training of future teachers and Concept Cartoons. Partially, these topics have been repeatedly discussed at SEMT conferences: formative assessment from the perspective of students was explored by Martin, Wang, Lambert, Polly and Pugalee (2015), with relation to inquiry based mathematics education by Hošpesová, Stuchlíková and Žlábková (2017), questions related to inquiry based education from the perspective of future teachers were reported by Hošpesová, Samková, Tichá and Roubíček (2015), Samková (2017), Concept Cartoons were introduced by Samková and Naylor (2015), Samková, Tichá and Hošpesová (2015).

Issues reported here are a part of a larger educational research project supported by the Technology Agency of the Czech Republic named *Learning Hyperspace for Formative Assessment and Inquiry Based Science Teaching*. The goal of the project is to create a learning hyperspace for teachers where they could learn how to implement formative assessment into their inquiry based teaching. This particular paper belongs to the preparatory stage of the project in which we map the properties of the connection between inquiry based education and formative assessment and their possible impact on the form and content of the hyperspace, and also look for appropriate tools that could be used as a part of the hyperspace.

The reported empirical study aims to address the following research question: “Which aspects related to on-the-fly formative assessment appear in future teachers' written responses to Concept Cartoons and in a subsequent discussion?”

**Inquiry based (mathematics) education**
Inquiry based pedagogy is usually characterized as a way of teaching in which students are invited to work in ways similar to how scientists work (Artigue and Blomhøj, 2013). These procedures are naturally adapted to school context, so that students do not discover new scientific issues but rediscover school mathematics or solve simple problems of everyday application character. The role of the teacher in such lessons consists mainly in creating a suitable learning environment, building upon students’ reasoning, giving students support, and in connecting to students’ experience (Dorrier and Maaß, 2014).
In mathematics, a suitable learning environment can be achieved through tasks that are open in the sense of open approach to mathematics (Samková, 2017), i.e. tasks that have multiple ways of grasping, multiple correct ways of solving, multiple correct answers and/or multiple ways of transforming the task into a new one (Nohda, 2000). The process of solving such tasks consists of various ways of formulating the task mathematically, various ways of interpreting the formulation, of investigating various approaches to the formulated task and the results found, of various ways of interpreting the results, and/or of posing various advanced tasks. Such an environment is rich in alternative reasoning, correct as well as incorrect, based on various expedient ideas (e.g. employing appropriate concepts, generalizing through genuine relations among the concepts, providing logical arguments) or various misconceptional ideas (e.g. employing inappropriate concepts, generalizing through false relations among appropriate concepts, providing illogical arguments).

**Formative assessment**

Formative assessment, as assessment for learning, comprises varied strategies that ought to be implemented into the process of teaching: clarifying and sharing learning intentions and criteria for success, engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding, providing feedback that moves learners forward, activating students as instructional resources for one another, and activating students as the owners of their own learning (Black and Wiliam, 2009). When performed in the classroom, formative assessment may take varied appearances: e.g. of on-the-fly assessment, structured assessment dialogue, peer-assessment, written feedback (Dolin and Evans, 2018).

This paper will address the on-the-fly appearance. In that case, the teacher spontaneously recognizes appropriate opportunities to support students in advancing their learning, and promptly induces a discussion in the classroom (Shavelson et al., 2008). That means that such assessment practice relies on the quality of teacher’s ability to notice, interpret and promote various classroom phenomena that are related to content knowledge of learners. From the perspective of inquiry based mathematics education and open tasks, the teacher ought to be able to recognize and fully understand various ways of grasping the tasks, various solution procedures and various results – not only the commonly used ones but also the uncommon ones (e.g. specific or innovative).

In relation to inquiry based education, scholars usually concur that formative assessment is essential while implementing inquiry based methods of teaching (Harlen, 2013; Dolin and Evans, 2018) and that when properly implemented, formative assessment naturally penetrates the process of inquiry (Hošpesová, 2018). On the other hand, many of future teachers and teachers have never experienced formative assessment as learners, nor have they been trained to implement it in their own teaching (Rokos and Zavodska, 2015).
Concept Cartoons

An educational tool called Concept Cartoons appeared in UK 30 years ago (Keogh and Naylor, 1993), and since then it has been implemented into science and mathematics education at many elementary schools there. Each Concept Cartoon is a simply picture of a bubble dialog among several children, discussing a situation familiar to them from school or everyday reality. Particular bubbles present various alternative opinions on the pictured situation, some of them correct, some incorrect, some might have their correctness unclear or conditioned (see Figures 1, 2, 3). When working with Concept Cartoons in the classroom, the teacher usually asks students to decide which of the pictured children are right and which are wrong, and calls the students for justification of their decisions. Such an arrangement enables the teacher to use Concept Cartoons as a valuable tool in formative assessment (Naylor and Keogh, 2007).

The Concept Cartoons tasks that have two or more correct bubbles can be also considered as open in the sense of open approach to mathematics, since their correct bubbles offer various ways of grasping, various ways of solving and/or various solutions to a mathematics-related problem (Samková and Tichá, 2016). Such a perspective aligns Concept Cartoons with inquiry based mathematics education.

![Concept Cartoon](image_url)

Figure 1: Concept Cartoon on the introductory to the topic of ratios; the template of children with empty bubbles taken from (Dabell, Keogh and Naylor, 2008, no. 2.10)
Figure 2: Concept Cartoon on the part-whole interpretation of fractions; the template of children with empty bubbles taken from (Dabell et al., 2008, no. 3.10)

Figure 3: Concept Cartoon on the introductory to the topic of inequalities; the template of children with empty bubbles taken from (Dabell et al., 2008, no. 1.3)
In this paper, I will render another insight into how Concept Cartoons might be used in relation to formative assessment and inquiry based mathematics education. In my previous research on using Concept Cartoons in university preparation of future primary school teachers, I started to consider each Concept Cartoon as an educational model of a classroom situation, as a representation of school practice that focuses on various content-related comments provided by students and on various ways how teachers may response to them (Samková and Hošpesová, 2016; Samková, 2018). In that sense, Concept Cartoons might help future teachers recognize and fully understand various ways of grasping, various solution procedures and various results that could have been provided by students, and thus create a basis for their own future realization of on-the-fly formative assessment.

As the tool in the study, I created a set of 10 Concept Cartoons related to various topics from primary school mathematics. Five of them were composed as completely new (e.g. Figures 1, 3), the other five were modified from the original set by Dabell et al. (2008) by changing some numbers in the assignment and/or changing the content of some bubbles (e.g. Figure 2). Each of the 10 Concept Cartoons could be considered as open: six of them were with multiple ways of grasping (e.g. Figure 1), nine with multiple correct ways of solving (e.g. Figure 2), three with multiple correct answers (e.g. Figure 3), and four with multiple follow-up tasks (e.g. Figure 3).

The study took place in two lessons in two consecutive weeks. At the first lesson, I assigned the set of Concept Cartoons to the participants, and asked them to respond to the Concept Cartoons in written form: decide which of the pictured children were right and which were wrong, and justify their decisions. They worked on the task individually. The participants were given all the necessary time, so that this stage lasted 70 minutes eventually.

Afterwards, I collected the responses and analysed them qualitatively, using open coding (Miles, Huberman and Saldaña, 2014). During data analysis, I focused on aspect related to on-the-fly formative assessment, i.e. to possible recognition and understanding of various ways of grasping, various solution procedures and various results.

At the second lesson, I returned the responses to the participants, and we started a detailed classroom discussion on the Concept Cartoons as well as on the responses. Again, the participants were given all the necessary time; this stage lasted 90 minutes. During the discussion, I wrote my filed notes on the content of the discussion, and added them to data for another round of qualitative analysis under the same conditions as before.

**Findings**

From the perspective of on-the-fly formative assessment, there were three noticeable aspects related to responses to the Concept Cartoons from Figures 1 to 3:
The A-bubble in Figure 1 might present an alternative grasping of the pictured situation, based on the idea of both the pictured scales occurring at once and on comparison of the weight of all pictured products (since the text in the bubble is in plural). In that case, the weight of all of the pictured lemons is twice the weight of all of the pictured aubergines and also twice the weight of all of the pictured bananas. This alternative idea was not revealed by any of the participants, though was widely discussed during the second lesson. The discussion led to many questions related to multiple grasping of tasks, i.e. to questions related to various types of ambiguous tasks and to various possible interpretations of their assignments. The participants asked me for other examples of tasks with multiple grasping, started to pose their own examples of situations or word problems that might be grasped variously, provided each other explanations why the posed examples were (not) duly formulated, were curious about possible occurrence of similar tasks in textbooks. Without any impulse on my side, the participants also started the discussion on how they could prepare properly for their own teaching in order to implement various ways of grasping in it, and on how often should they use such tasks in their teaching.

The bubbles B, C, E in Figure 2 present three different solution procedures leading to determining 3/4 of 12. Most of the participants revealed the content of the C-bubble which consists of the most common procedure used in our country, establishing 3/4 of 12 as 3-times 1/4 of 12. They also revealed the content of the E-bubble which establishes 3/4 via subtracting 1/4 from the whole. But almost all of them failed to reveal the content of the B-bubble which establishes 3/4 of 12 as 1/4 of 3-times 12. Unlike for instance in Germany (Krauthausen and Scherer, 2002), this particular concept is not common in our country. The discussion of this alternative led to multiple questions about various interpretations of fractions, graphical representations, various practically-based word tasks on fractions. Many of the participants commented the B-bubble in the sense that “the result is correct but I do not understand the procedure”, so that the discussion also led to a substantial debate whether the correctness of a result is enough to declare as correct a bubble which shows an unknown procedure leading to this result.

With the Concept Cartoon from Figure 3, there was no particular bubble that would trigger a debate, but the task as a whole. During the discussion, interesting follow-up questions appeared on how heavy is the flower compared to the cube, whether the number of items on one side of the seesaw is relevant for determining the heaviest/lightest item, or whether the task would be easier with both seesaws (un)balanced.

In summary for all 10 Concept Cartoons, it might be stated that as in the above paragraphs, the issues that triggered the debate were not usually related to a
particular mathematical content (ratio, calculation algorithm, area, data handling, etc.) but rather to the openness of the task. From the perspective of participants, it might be stated that all of the participants took part in the discussion and that they contributed to the discussion more or less regularly and evenly.

Discussion

The findings of this small study enriched the puzzle on “How can we meaningfully employ Concept Cartoons in future teacher education” by another piece of knowledge. They show how Concept Cartoons might be used in preparation of future teachers for better recognizing and understanding of various ways of grasping, various ways of solving and various results, i.e. for supporting future implementation of on-the-fly formative assessment into their school practice. Such an arrangement also effortlessly interconnects formative assessment with inquiry based mathematics education, since suitably created Concept Cartoons can be considered as open tasks and therefore also as inquiry tasks.

In a more general perspective, the text also addresses the issue of various possible formats that representations of school practice may have, namely the formats that can be regarded as a result of a decomposition of practice according to Grossman et al. (2009). Such decompositions involve breaking down practice into its constituent parts for the purposes of teaching and learning. In the particular decomposition consisting of the tool based on Concept Cartoons, the constituent part focuses on various content-related comments provided by students and on various ways how teachers may response to them.

The issues addressed in subsequent discussions on Concept Cartoons relate to the issues that are usually described as the troubled ones from the perspective of teachers (Biton, Hershkovitz, Hoch, Ben-David and Fellus, 2017): dealing with assessment on method as opposed to assessment on results, dealing with missing description of the thinking process and dealing with difficulties in seeing the thinking process that led to the solution. They are also closely related to noticing skills and knowledge based reasoning (Vondrová, 2018).

As for the format of the Concept Cartoon tool itself and the format I propose for its usage in future teacher education, there are two matters that should be addressed here. First, the picture-based environment is not common in research on mathematics education issues and therefore is sometimes considered as unsuitable since the pictures are understood as disturbing and distracting. In this context, there have been interesting recent findings provided by Herbst and Kosko (2013), Friesen and Kuntze (2018) who made a comparison of various formats of representations of school practice (classroom videos, staged videos, animations, comics, vignettes and transcripts) assigned to teachers and future teachers in order to investigate their pedagogical content knowledge. Data analysis showed no significant differences between the formats in relation to the quality of the responses a well as in relation to the difficulty of the task. Second, unlike the
original classroom-discussion format of Concept Cartoons given by Keogh and Naylor (1993), I employ the tool individually and in written format. It appears that with future teachers this new format is valuable, by providing a wide range of information on their pedagogical content knowledge in the written responses (Samková, 2018) or by eliciting subsequent discussions on the written responses (here). Such written format is similar to the one used by Martin et al. (2015) to support formative assessment and learning of mathematics through writing.

The three Concept Cartoons tasks that were discussed in detail in this paper, all belong to practically based problems. Such tasks are a natural source of open and inquiry tasks for mathematics education, since they often have multiple ways of grasping and multiple ways of interpreting the results; sometimes also the classification of the results can be unclear, biased, difficult or even impossible (Koman and Tichá, 1998). Similar properties have also so called ill-structured problems (Fielding-Wells, Dole and Makar, 2014).

Acknowledgement: This research was supported by the Technology Agency of the Czech Republic under Grant ‘Learning Hyperspace for Formative Assessment and Inquiry Based Science Teaching’, project No. TL02000368.

References


INCLUSIVE MATHEMATICS – IN-SERVICE TRAINING FOR OUT-OF-FIELD TEACHERS

Petra Scherer, Marcus Nührenbörger and Leonie Ratte

Abstract

Coping with heterogeneity in mathematics classrooms is a big challenge for teachers. Particularly, considering students with special needs in inclusive classrooms requires further developments of classroom instruction. For teachers’ professional development, in-service courses for mathematics are necessary for both professions, for special education teachers as well as for teachers in regular schools. As many special education teachers are out-of-field teachers for mathematics, a specific program for laying a mathematical and didactical foundation is advisable. In the paper the design of a concrete in-service course is sketched, and corresponding experiences and findings are discussed.

Keywords: in-service teacher education, inclusive mathematics, special education

Introduction

In Germany, about 50% students with special needs on the primary level visit regular schools in inclusive settings (cf. Klemm, 2018). Inclusive classrooms

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show extremely heterogeneous groups, and the teacher is confronted with individual handicaps, for example deficits in language or visual perception, failure of concentration or reduced memory, so that a high degree of differentiation is needed. Research shows that for successful inclusive subject-matter instruction – beyond teachers’ fundamental attitudes and beliefs – not only teachers’ competences with respect to special education are of great importance but also with respect to the subject, here mathematics (Gasterstädt and Urban, 2016; Heinrich et al., 2013). In Germany, many special education teachers teaching in inclusive settings are out-of-field teachers for mathematics: They plan and teach mathematics lessons without having a university qualification for mathematics or mathematics education. Moreover, until now, most teacher education programs for special education on the one hand, and for the regular school system on the other hand, follow different concepts and underlying paradigms for teaching and learning (cf. Scherer, 2019a). Consequently, in-service courses are of great importance to qualify out-of-field teachers in mathematics, but there does not exist an official obligation for attending in-service trainings in Germany. In the German Center for Teacher Education Mathematics (Deutsches Zentrum für Lehrerbildung Mathematik – DZLM; dzlm.de) a course was developed addressing out-of-field teachers working in inclusive mathematics classrooms (cf. Scherer and Nührenbörger, 2017; Scherer, 2019b). In the following the design of a concrete in-service course will be sketched, and corresponding experiences and findings will be discussed.

**In-Service Course Concept for Out-of-field Teachers in Inclusive Mathematics**

Addressing the aforementioned in-service requirements, the DZLM developed an in-service course for special education teachers who were out-of-field teachers for mathematics on the primary level. The course was hold in cooperation with local governments in the school year 2015/16 and repeated two times in the school years 2016/17 and 2017/18 (cf. Scherer and Nührenbörger, 2017). It will be offered again during the school year 2019/20.

The course aimed at deepening the teachers’ mathematical and didactical competences and focused also on the reflection of teaching and learning processes. It should enable the special education teachers to act flexibly, adaptively and mathematically correct in inclusive settings. In detail, the participants should

- acquire central mathematical and didactical competences with respect to central topics of primary mathematics, to be able to analyse mathematical tasks and problems as well as students’ documents correctly,
- get to know central didactical principles for planning inclusive mathematics and suitable activities for students with and without learning difficulties, try out those settings and reflect on their experiences,
become acquainted with typical mistakes as well with diagnostic instruments for regular classroom situations and test those methods and instruments in their classrooms.

In a research-based design-approach the concept focused on central ideas of a basic mathematical education as formulated in the standards and the curriculum for primary mathematics (cf. KMK, 2005; MSW, 2008). The acceptance and consideration of students’ individual needs as well as support of all students was one of the core ideas (cf. Häsel-Weide and Nührenbörger, 2017, 8f). The concrete course topics will be presented in the section Course Contents.

The course was planned for five course days, distributed over a period of five months during the school year. The concept considered the following six DZLM-design principles for in-service teacher training (cf. DZLM, 2015; Barzel et al., 2018): ‘Competence-oriented’, ‘Participant-oriented’, ‘Diverse instruction formats’, ‘Case-relatedness’, ‘Stimulate cooperation’, and ‘Foster (self)reflection’. Those principles represent the national and international state-of-the-art research findings and can be regarded as quality factors for effective in-service training (cf. Lipowsky and Rzejak, 2015; Timperley et al., 2007; Amrhein and Badstieber, 2013).

In total, the course covered 80 hours, and beyond attending the course days, the teachers had to work on activities through self-study, do homework and try out tasks and activities in their classrooms.

The accompanying research comprised a pre-post-questionnaire that asked for the participants’ attitudes and beliefs with respect to inclusion in general and especially for inclusive mathematics (cf. Meyer, 2011). Additionally, the post-questionnaire at the end of the course included some content-related items asking for a retrospective self-assessment of their own competence development (see the section Evaluation Results). These items were designed according to the course topics and corresponding objectives.

**Course Contents**

Each in-service day was related to a central topic relevant for primary mathematics, and the five topics were the following: ‘Objectives and Suitable Learning Offers for Inclusive Mathematics’, ‘Number Representations and Imaginations’, ‘Imaginations of Operations’, ‘Mathematics in Contexts’, and ‘Diagnosis and Support’. With all these topics, the participants should realize the specific meaning of process oriented competences for learning mathematics (cf. MSW, 2008, 57 f.) and the potential of substantial learning offers (cf. Wittmann, 2001). In the following, a short description of the design of each day will be given:

*Day 1 – Objectives and suitable learning offers for inclusive mathematics*

The first day took up the teachers’ general pedagogical knowledge and special education knowledge with respect to good learning arrangements, and initiated the transfer to suitable learning arrangements in (inclusive) mathematics.
Common learning on common topics was stressed, without neglecting the work on individual topics and objectives or denying additional support for students (cf. Häsel-Weide and Nührenbörger, 2017, p. 15). Following the concept of a natural differentiation, learning arrangements should be offered that address the heterogeneous groups and individual learning in an inclusive classroom (cf. Scherer, 2015, p. 270). The out-of-field teachers were given the opportunity to try out selected formats like ‘Discover Patterns and Structures’ or ‘Number Triangles’ by themselves, analyse different levels of strategies and develop adaptive ideas for their own students (cf. Krauthausen and Scherer, 2013).

**Day 2 – Number Representations and Imaginations**

For the second in-service day the participants had to examine the relevance of suitable imaginations for numbers and operations, a central objective for arithmetic in primary school (cf. MSW, 2008, p. 58). Numbers can be represented in different ways and on different levels, they are related to each other, contain specific characteristics and properties and show a variety of aspects. For supporting students in building up suitable imaginations, the teacher has to offer suitable representations and manipulatives, and a well-directed support requires sufficient content knowledge in this field.

**Day 3 – Imaginations of Operations**

Taking up the topic of day 2, the participants should deepen their knowledge with respect to the imagination of operations. Primary students should develop adequate imaginations for the central operations addition, subtraction, multiplication, and division (cf. MSW, 2008, p. 61).

**Day 4 – Mathematics in contexts**

The fourth day discussed central objectives and functions for context-embedded mathematics and drew the teachers’ attention to the complexity of modelling processes (cf. Scherer, 2009). Selection and use of context problems is dependent from the underlying objective and function and has to be reflected carefully by the teacher. Embedding mathematics in contexts might be help or hindrance for the students (cf. van den Heuvel-Panhuizen, 2005), and the teacher has to be aware of the complex demands of solving context problems.

**Day 5 – Diagnosis and Support**

This topic represents a central topic for special education but might be restricted to general aspects and does not cover subject specific knowledge and expertise. According to the design principle ‘participant-oriented’ the teachers’ existing knowledge was taken up and extended by content-related elements. This topic will be described more detailed in the section Exemplary Activity “Diagnosis in Mathematics”.

The second part of this in-service day was devoted to the topic of productive support. The participants got to know main principles of productive support in mathematics classroom, and it was pointed out that there is no need for a special
didactic for individual students. Rather, it was stressed – connected to the first in-service day – to orientate on the central elements for a basic mathematical education that should be offered to all (cf. Häsel-Weide and Nührenbörger, 2017, p. 10).

The whole course aims at making the teachers more sensitive not only for the variety and usefulness of different ways of thinking and strategies, but for the realization that students with special needs do not need specific didactics. Those students are in need of teachers who are able to analyse their learning processes correctly and deeply (cf. Häsel-Weide and Nührenbörger, 2017, p. 9). With these insights, the teachers might be able to organize and accompany common learning situations for all students, enabling individual as well as cooperative learning situations.

**Exemplary Activity “Diagnosis in Mathematics”**

In the following section, selected activities for the topic ‘Diagnosis and Support’ will be presented more detailed and connected to the design principles.

At the end of in-service day 4 the participants got a preparatory task they had to try out at school: They should offer the representations in Figure 1 to at least two low achievers at school and let the students work out the number of dots. The teachers should observe and analyse the different strategies, errors and difficulties. They should also reflect on the demands comparing the numbers 37 and 48 and document their experiences. The activity aimed at understanding the variety of possible students’ strategies, the possible structures of this representation and at fostering the awareness of possible difficulties and errors. With this activity the design principles ‘participant-oriented’, ‘case-relatedness’ and ‘foster (self)reflection’ were in the focus.

![Figure 1: Task for working out a number of dots (cf. Scherer, 1997)](image)

The in-service day started with an input concerning different diagnostic methods (in the range from standardized to informal procedures) and a discussion about
the specific value or the limitations of a method and procedure. Afterwards the main stress was put on qualitative methods for analysing students’ thinking and difficulties, followed by the reflection of the tested task (Figure 1). The teachers’ reports were discussed and the participants should exchange their experiences (stimulate cooperation). Additionally, video documents showing students with special needs working on this task (cf. Scherer, 1997) were shown and the participants could relate those cases to their own experiences (diverse instruction formats). As a further diagnostic method the analysis of students’ written documents was discussed and practiced by the participants.

For deepening the topic and the own competences, as a follow-up task the teachers should analyse items used in large-scale studies and analyse different errors in students’ documents with written multiplications (cf. Hoffmann and Scherer, 2016).

**Evaluation Results**

The first try-out of the course was attended by 18 special education teachers, and the final evaluation showed that the participants assessed the course in general good or very good (scale from 1 to 5 from very good to unsatisfactory).

The participants agreed to the following statements that refer to the realization of the DZLM-design principles (Likert scale from 1 to 4; 1 = not true; 4 = true):

- **Diverse instruction formats**: The course methods were matched well to the content.

- **Case-relatedness**: The relevance of the theoretical aspects was illustrated by examples of practice. I had sufficient opportunities to contribute examples taken from my own practice.

- **Competence-oriented**: I could try out my new competences by active participation.

- **Stimulate cooperation & Foster (self)reflection**: I had sufficient opportunities to work on tasks and problems together with other participants.

Retrospective self-assessment can be regarded as a valid method (Nimon et al., 2011), and at the end of the course the teachers had to rate their competencies for selected aspects on the one hand in general, on the other hand concerning the relevance for inclusive mathematics (Figure 2).

These self-assessments do not surprise: The richness of aspects of numbers as well as a deepened reflection on different number representations for mathematics instruction did not belong to the secure knowledge of the out-of-field teachers before the course (see also Jandl and Moser Opitz, 2017), and appear to have developed during the course. Moreover, the relevance for inclusive settings especially for students with special needs is now stated explicitly.
I know the relevance of viable number imaginations and different possibilities for representing numbers with material and manipulatives.

Change of representation and building up flexible number imaginations appear relevant to me for inclusive mathematics and for supporting students with special needs.

Figure 2: Retrospective self-assessment of one’s own competence development for the topic ‘Number Representations and Imaginations’ (N = 17)

I know informal diagnostic approaches for differentiated grasping of mathematical competences and know measures for supporting basic competences.

Diagnostic interviews and differentiated support of basic competences appear relevant to me for inclusive mathematics and for supporting students with special needs.

Figure 3: Retrospective self-assessment of one’s own competence development for the topic, Diagnosis and Support’ (N = 17)
Compared to these findings, the self-assessment for the topic ‘Diagnosis and Support’ shows a slighter development (Figure 3).

But this result could be expected as this topic represents a more or less classical field of expertise for special education. So, one can expect that the special education teachers already had a high expertise before the course started. Nevertheless, the participants notice a strong development of their competences and also see a higher relevance for teaching. Not least, this might be attributed to the fact that the diagnostic competences the teachers had before were not subject-specific. It might also be that their earlier competences were related, more or less, to standardized methods and instruments, whereas the in-service course concentrated on diagnostic approaches that showed a close connection to everyday classroom situations.

On the basis of this evaluation the further development of the course concept was done: A stronger cooperation of the participants with their primary teachers at their own schools was stimulated, for example through follow-up tasks after a course-day (For lesson planning: ‘Which different perspectives or knowledge will you bring in? Who is responsible for what task?’ For lesson analyses: ‘Analyse your students’ documents with respect to underlying imaginations of operations. Take the chance of exchange with your colleagues at school.’). Moreover, the individual topics were modified lightly and the order of topics was changed. As the participants asked for, the topic ‘Informal Calculations and Written Algorithms’ was added while shortening the topics ‘Mathematics in Contexts’ and ‘Imaginations of Operations’ at the same time.

Conclusions

Content specific knowledge is a necessary prerequisite for teaching mathematics or other subjects. Focusing on out-of-field teachers for inclusive mathematics classrooms, analyses and reflections on materials, videos and examples given in the course have a high value. Extended by the teachers’ own experiences and common reflections during the course sessions can increase their knowledge and teaching repertoire for the future.

Realizing the UN Conventions (see UN, 2006) in school requires adequate teacher education programs and integrating this topic in various in-service as well as preservice courses will be necessary (see also Scherer, 2015).

Acknowledgement: The project DZLM is supported by Deutsche Telekom Stiftung (DTS).

References


This article describes part of a study that explored two factors which affect teaching-learning processes: teachers’ knowledge (common content knowledge, specialized content knowledge, knowledge of content and teaching, knowledge of content and students) and their self-efficacy regarding these components of knowledge. These factors were explored among 64 teachers for learning-disabled students, who teach multiplication and division. The research findings indicated that the teachers' knowledge in the various knowledge types was lacking, especially regarding the knowledge of content and students. The findings also demonstrated that there were differences between teachers’ level of self-efficacy related to the various knowledge components. The lowest self-efficacy level was found regarding pedagogical knowledge of content and students.

Keywords: content knowledge, pedagogical content knowledge, self-efficacy, special education, multiplication, division
and division of natural numbers and zero: an important topic in mathematics curricula.

Theoretical Background

Knowledge required for mathematics teaching

Three of the components of knowledge needed for teaching are subject matter knowledge, curricular knowledge and pedagogical content knowledge (Shulman, 1986). Researchers have tried defining more accurately these terms with reference to mathematics (Ball et al., 2008) and classified two components of subject-matter knowledge (common content knowledge and specialized content knowledge) and two components of pedagogical content knowledge (pedagogical knowledge of content and teaching and pedagogical knowledge of content and students).

Common Content Knowledge (CCK) is a type of mathematical knowledge required also by those who do not teach, e.g. knowledge of solving or calculating. Specialized Content Knowledge (SCK) is mathematical knowledge and competences that are unique for teaching. For example, solving a problem in various ways and examining different representations for the same problem. Knowledge of Content and Teaching (KCT) is a combination of subject-matter knowledge and teaching. For example, which examples are suitable for presenting a topic, which examples should be used in order to enhance the learnt content, assessing the advantages and disadvantages of various tasks as well as being acquainted with different methods of representing a problem. Knowledge of Content and Students (KCS) is the integration of subject-matter content with acquaintance of students. For instance, knowing students’ common errors, recognizing possible reasons for these errors and identifying what is easy/difficult for a certain learner population.

A study that explored teachers’ mathematical content knowledge and pedagogical content knowledge found that the wider the teachers’ knowledge was, the more extensive the students’ knowledge and the better their attainments were (Tchoshanov, 2011). A study that investigated knowledge components of special education mathematics teachers illustrated that the teaching quality of teachers in special education classes is associated with their knowledge of the discipline (Bronwell et al., 2010): the greater knowledge the teachers have – the better they cope with students’ learning disabilities and the more effective intervention programs they prepare.

Self-efficacy

In addition to teachers’ knowledge, there is another factor that might affect the quality of the teaching-learning process, self-efficacy. In order to perform a task effectively, people need both the suitable skills and the belief and confidence in their ability to apply them as required (Bandura, 1977). Self-efficacy may affect people’s functioning: the higher people’s self-efficacy, the more they persevere
in efforts and actions. Self-efficacy is a factor that might affect the teaching-learning process, since the functioning of both teachers and students in class may be related to the level of confidence of each of them in the ability to successfully fulfill their role (Dellinger, Bobbett, Olivier and Ellett, 2008). Teachers whose level of self-efficacy was high had greater job satisfaction, were more involved in the preparation of personal curricula for the students and cooperated better with the parents and colleagues (Brouwers and Tomic, 2000).

Regarding self-efficacy of teachers in special education classes, studies illustrate that teachers with a high level of self-efficacy are more likely to try different ways of teaching, to be organized in their instruction, to have better relations with students and to be confident and enthusiastic about teaching (Allinder, 1994). In addition, self-efficacy was found to be negatively correlated with burnout, suggesting that the higher their self-efficacy levels are, the better special education teachers function over time (Sarıçam and Sakız, 2014).

**Multiplication and division (of natural numbers and zero)**

Multiplication and division of natural numbers are a central part of the mathematics curricula in Israel and in other countries. The mathematics curriculum in Israel, designed for mainstream education and special education, emphasizes that in the process of teaching multiplication and division, teachers need to develop computational competences as well as encourage numerical insight manifested, among others, by different solution strategies and comprehension of different solution methods (Ministry of Education in Israel, 2006).

Researchers underscore the importance of teachers’ pedagogical knowledge. For example, a study showed that teachers’ teaching methods had a relation to the learners’ attainments (Ma, 1999). The students were more successful when they studied with teachers who had wider knowledge, who taught in a way that made students understand the meaning of the algorithms rather than in a merely technical manner. Another study showed that when teachers could explain the underlying principle and the meaning of the division algorithm, the students demonstrated a more thorough knowledge, created relations between operations and terms and their responses were more varied (Takker and Subramaniam, 2018).

Various studies (e.g. Lee, 2007) indicated the two components of pedagogical content knowledge that teachers need, knowledge of content and teaching and knowledge of content and students. As for knowledge of content and teaching, researchers recommend multiplication and division teaching methods, emphasizing the importance of consolidating the meanings of multiplication and division in the teaching process by using illustrations, solving with distributive property, using the rectangle area model for multiplication, everyday problems and games (Boaler, 2015; Cimen, 2014; Lee, 2007). Other researchers suggest ways of presenting and teaching the topic of division with various models for
promoting the understanding of this issue (Jong and Magruder, 2014). Specifically, recommendations for teaching students with learning disabilities emphasize the importance of using demonstrations, concrete elements and connections to everyday life in order to reinforce understanding over algorithms (Bakker, Heuvel-Panhuizen and Robitzsch, 2016; Milton, Flores, Moore, Taylor and Burton, 2019). As for knowledge of content and students, researchers who examined and characterized students’ common errors in multiplication and division noted misunderstanding the base-ten number composition and algorithmic errors (Bainbridge, 1981; Radatz, 1979). As for self-efficacy, most studies related to teaching learning-disabled students investigated teachers' general self-efficacy for teaching (Sarçam and Sakız, 2014) and not the self-efficacy related to knowledge of multiplication and division.

The aims of the present study are: a. to explore teachers’ knowledge related to teaching multiplication and division. b. to explore teachers’ level of self-efficacy related to their knowledge.

**Methodology**

The research participants were 64 Israeli mathematics teachers who teach multiplication and division of natural numbers in special education classes for learning-disabled students. All the teachers have a Bachelor of Education degree in special education with between 1-35 years of experience. They were trained in different types of mathematics teaching frameworks: 29 teachers specialized in mathematics teaching as part of their academic studies; 19 teachers were trained in mathematics teaching within instruction frameworks and in-service training programs; and 16 teachers did not receive any special training for mathematics teaching.

The research instruments were two questionnaires, a Knowledge Questionnaire and a Self-Efficacy Questionnaire, that were validated by three mathematics teaching experts – researchers of mathematics education in elementary school and in special education. A preliminary study was conducted with 12 teachers and reliability was examined. A Cronbach’s coefficient alpha was calculated, using the pilot data. The reliability score (> 0.7) of the instruments had an acceptable level of reliability.

The Knowledge Questionnaire consisted of 24 open-ended items (involving computations less than 100, multiplying/dividing by whole tens and computations greater than 100). It examined teachers’ knowledge of the four knowledge components defined by Ball et al. (2008) in the following manner: For examining CCK, the teachers were asked to solve multiplication and division exercises. For examining SCK, the teachers were requested to solve an exercise in another way or to estimate the result of the exercise without solving it. For examining KCT, the teachers were asked to present a way of teaching or illustrating an exercise. For examining KCS, the teachers were requested to describe typical errors made
by students when solving an exercise as well as to explain the reasons for such typical errors.

The Self-Efficacy Questionnaire comprised of 24 statements, which the teachers ranked on a 1-5 scale, according to their level of confidence in performing what the statement indicated (5 = very confident and 1 = not confident at all). For each item in the Knowledge Questionnaire there was a matching statement in the Self-Efficacy Questionnaire (see Table 1).

In this article we analyze the results of one multiplication exercise and one division exercise. In table 1 you can see the items related to the multiplication exercise.

<table>
<thead>
<tr>
<th>Knowledge questionnaire</th>
<th>Self-efficacy questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CCK</strong> Solve 29 x 58 in solving multiplication exercises involving computations greater than 100</td>
<td><strong>“What is the level of your confidence ...”</strong></td>
</tr>
<tr>
<td><strong>SCK</strong> Solve the exercise 29 x 58 in another way in solving multiplication exercises (involving computations greater than 100) in more than one way</td>
<td></td>
</tr>
<tr>
<td><strong>KCT</strong> Present two ways of teaching the exercise 29 x 58 teaching the multiplication algorithm</td>
<td></td>
</tr>
<tr>
<td><strong>KCS</strong> Present two errors that you think students will make predicting students’ errors in solving multiplication exercises (involving computations greater than 100)</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Examples of items from the two questionnaires

**Research Findings**

The results analyzed below are of four items of knowledge and four matching self-efficacy statements related to the multiplication exercise 29 x 58, and of four items of knowledge and four related self-efficacy statements related to the division exercise 1400 : 7.

**Findings related to multiplication**

The following findings relate to the multiplication exercise 29 x 58. The teachers were asked four questions designed to explore their knowledge of the four knowledge components (Table 2). The findings in Table 2 illustrate differences in the teachers’ knowledge of the various components, with reference to this question.
### Table 2: Percentage of appropriate answers to the knowledge questions with reference to the exercise 29x58

The results shown in table 2 illustrate that in general the teachers have greater content knowledge (both common and specialized content knowledge) than pedagogical content knowledge. Regarding the differences between CCK and SCK a third of the teachers did not know how to solve the exercise in another way. Regarding pedagogical content knowledge, the teachers' knowledge was lacking: only 20%-25% knew other ways of teaching or common errors.

The teachers’ self-efficacy regarding their knowledge is presented in table 3.

### Table 3: Percentage of teachers who ranked every confidence level, mean and standard deviation

Table 3 illustrates that, like the findings related to knowledge, the teachers were less confident in their knowledge of teaching and of students than in their content knowledge. The teachers generally ranked their confidence levels high, between 3-5. However, their confidence levels in their pedagogical content knowledge
were lower than their confidence levels in their content knowledge, and the variance of the findings of pedagogical content knowledge was greater.

**Findings related to division**

The following findings relate to the division exercise 1407 : 7. The teachers were asked four questions designed to examine their knowledge of the four knowledge components. The findings illustrate differences in the teachers’ level of knowledge of the various components with reference to this question (Table 4).

<table>
<thead>
<tr>
<th>Question</th>
<th>Type of knowledge</th>
<th>Percentage of appropriate answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solve the exercise</td>
<td>CCK</td>
<td>94%</td>
</tr>
<tr>
<td>Solve the exercise in another way</td>
<td>SCK</td>
<td>61%</td>
</tr>
<tr>
<td>Present two ways of teaching the exercise</td>
<td>KCT</td>
<td>Presented two ways 25%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Presented one way 53%</td>
</tr>
<tr>
<td>Present two errors that you think students will make</td>
<td>KCS</td>
<td>Presented two errors 6%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Presented one error 60%</td>
</tr>
</tbody>
</table>

Table 4: Percentage of appropriate answers to the knowledge questions with reference to the exercise 1407 : 7

Table 4 indicates that, like the findings related to multiplication, the teachers’ knowledge differs between the various components of knowledge. The greater knowledge was found in the common content knowledge. Pedagogical content knowledge was found lacking, especially knowledge of content and students.

The teachers’ self-efficacy regarding their knowledge is presented in Table 5. The findings show that, like the knowledge findings (Table 4) and like the multiplication findings (Table 3), the confidence levels of the teachers in the pedagogical content knowledge were lower than their confidence levels in their content knowledge and the variance of the findings of pedagogical content knowledge was greater.

<table>
<thead>
<tr>
<th>Assertion: The level of confidence to…</th>
<th>Type of knowledge</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>M(Sd)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solve division exercises</td>
<td>CCK</td>
<td>64</td>
<td>19</td>
<td>12</td>
<td>5</td>
<td>0</td>
<td>4.43</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.88)</td>
</tr>
<tr>
<td>Solve division exercises in more than one way</td>
<td>SCK</td>
<td>58</td>
<td>19</td>
<td>16</td>
<td>6</td>
<td>1</td>
<td>4.27</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(1.05)</td>
</tr>
</tbody>
</table>
Teach the algorithm of division | KCT | 48 | 33 | 9 | 8 | 2 | 4.17 (1.67)
Predict students’ errors in division exercises | KCS | 30 | 36 | 23 | 8 | 3 | 3.82 (1.99)

Table 5: Percentage of teachers who ranked every confidence level, mean and standard deviation

Discussion and Conclusions

The present study explored content knowledge and pedagogical content knowledge of mathematics teachers in special education classes as well as their self-efficacy related to their knowledge. Several meaningful findings were obtained from the study, the prominent among them being the differences in the knowledge the teachers demonstrated in the various knowledge components. The greater knowledge the teachers demonstrated was of CCK and the least knowledge the teachers demonstrated was of KCS. That is, most of the teachers could solve the exercise well, whereas some of them encountered difficulties in finding an alternative way of solving that exercise, some could not illustrate the exercise and about half of them found it difficult to predict students’ typical errors and indicate the source of these errors.

The gaps in knowledge manifested by the teachers regarding SCK and KCT imply their difficulties in teaching the arithmetic operations of multiplication and division. They found it hard to present teaching methods except for an algorithmic way as well as to suggest ways of presenting the exercises. A possible reason for these results is the fact that 16 teachers were not specialized in mathematics teaching. The gaps in knowledge were particularly prominent as far as KCS was concerned – the teachers were unable to indicate students’ typical errors. This may be because the teachers teach in classes of learning-disabled students, in which the students have varied characteristics and some of their errors are related to their learning disability. Another reason for this is the fact that among the participants were novice teachers with less than 5 years of experience, who are not acquainted enough with students' conceptions and misconceptions. This study was not about a comparison of teacher groups, and it is suggested to be studied in future research. However, teachers who teach in special education classes are varied in their experience and training, so we found it important to include them in the sample.

Another meaningful finding was the relationship between teachers' various knowledge components and their related self-efficacy. Correspondence between teachers’ level of confidence and their knowledge was found in the case of two knowledge components: common content knowledge (greater knowledge and high level of confidence) and knowledge of content and students (least knowledge and low level of confidence). Conversely, in the case of the other two knowledge components, no correspondence was found. The teachers demonstrated a high
level of confidence in their ability to solve multiplication and division exercises in more than one way and in their ability to teach such exercises. However, in the knowledge questionnaire, the findings did not align with the level of confidence and the percentage of appropriate answers was much lower.

Like Van Inger et al. (2016), we suggest counseling and guiding mathematics teachers for children with learning disabilities regarding various ways of teaching and illustrating multiplication and division. The findings of the current study could create an infrastructure for building intervention programs, aiming to promote the various knowledge components. Such programs could also increase the sense of self-efficacy among special education teachers who teach multiplication and division of non-negative whole numbers.

References


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**THE LINK BETWEEN CLASSROOM DISCOURSE AND STUDENTS’ AGENCY**

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**Abstract**

Mathematics lessons are highly criticized for being taught traditionally and do not promote explorative learning. In Israel, over the past decade, many attempts have been made to implement the theme of problem solving through pedagogical development (PD). This theme is based on the lesson study model for conducting mathematics lessons via an explorative approach. The present study observed eleven lessons after teachers’ participation in PD, with an emphasis on explorative instruction-learning process through watching and analyzing videoed pedagogical situations. This article is based on an analysis of videoed post-intervention lessons and focuses on the mathematical discourses, the cognitive demanding (support for learners), and students’ agency. A link
was found between the cognitive demands, the classroom discourse, and students’ agency.

Keywords: professional development, explorative instruction, mathematics teachers, classroom discourse, agency

Introduction

NCTM (2014) suggests that effective mathematics teaching practices inspire teachers to: implement tasks that promote reasoning and problem solving, use and connect mathematical representations, facilitate meaningful mathematical discourse, pose purposeful questions, build procedural fluency from conceptual understanding, support productive struggle in learning mathematics, and elicit and use evidence of student thinking (Chapman, 2016). Studies tend to focus and provide insight on one or two of these aspects of practice and succeeds in changing thinking and/or practice of them. However, it is also incumbent upon us to understand the link between these aspects.

This article exposes the link between high cognitively demanding task, classroom discourse and students’ agency.

Theoretical background

Teachers’ change, or pedagogical growth as an outcome of pedagogical development (PD) has been an extensive area of study in the last few decades (e.g., Kennedy, 2016; Sztajn, Borko and Smith, 2017). In the latest decades, various frameworks have been developed for advanced PD. These frameworks have a clear purpose, which is to change teachers’ practices to a more dialogic and problem-solving based instruction (Boston and Smith, 2011). Often, teachers do not adopt the practices as planned by the PD or “reform” leaders (Santagata, Kersting, Givvin and Stigler, 2011), and when they do, the adoption is accomplished in various stages of accuracy concurrent with the original ideas of the PD leaders (Schifter and Simon, 1992). Therefore, it leads towards the conclusion that change in teachers' practices is multifaceted and not linear (Heyd-Metzuyanim, Smith, Bill and Resnick, 2018).

The problem-solving approach has been proven to be effective for learners, and integrating this approach with pedagogy based on Lesson Study has helped teachers to generate a meaningful teaching-learning process (Groves, 2013; Takahashi, 2006). In this process, teachers are required to design lessons built vis-à-vis inquiry of a central mathematical problem that promotes students’ conceptual understanding (Groves, 2013). This type of teaching is called ‘explorative instruction’ (Heyd-Metzuyanim, Tabach and Nachlieli, 2016) and hence represents a remarkable shift from traditional teaching that is based mostly on procedures and practices.

Heyd-Metzuyanim, et al., (2016) claim that explorative instruction offers students to participate in explorative participation. The preoccupation with solving
a central problem promotes explorative participation as well as the construction of mathematical concepts and the relationships between them and procedures.

Hiebert and Grouws, (2007) claim that students’ opportunities to struggle with high cognitively demanding tasks and pay explicit attention to concepts are essential dimensions for students' conceptual understanding. The teacher’s role is to create learning opportunities that support students’ conceptual understanding through problem solving, as well as supporting student agency and authority (Schoenfeld, 2014).

Agency and authority is described by Schoenfeld (2014) as ‘the extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another’s ideas in ways that contribute to their development of agency (the capacity and willingness to engage mathematically) and authority, e.g., recognition for being mathematically solid’ (p.407). If this is so, agency and authority rely on discourse.

Discourse is the teacher's central pedagogical tool in class. Teaching is an interactive process, and classroom discourse is the mechanism through which this process takes place. The essence of each lesson, its components and its various forms, involves discourse (Cazden, 2010). Cazden (2001) found that discourse in the classroom is characterized by a repetitive narrative that includes three stages: the teacher poses a question, the student responds to the question, and the teacher evaluates the student’s response. This pattern is called IRE (Initiation-Response-Evaluation).

While other studies have linked professional development to changes in teachers’ knowledge, beliefs, or habits of practice, (Santagata, et al., 2011; Sztajn, et al., 2017) a critical question to consider is whether changes by implementing high demanding tasks had influence or maybe some effects on other practices.

This leads to the central issue of this study: Is there a link between teaching of high cognitively demands tasks, the mathematical discourse in classroom and the agency of students?

**Method**

The study consisted of 11 teachers who were certified to teach mathematics in elementary school and participants in the PD program that is centered on designing and implementing lesson plans according to the problem-solving approach (Hino, 2015). During the program, teachers engaged in high cognitive mathematical tasks, viewed videoed lessons demonstrating the problem-solving approach, discussed the pedagogical gains and mathematical ideas that emerged from those lessons, and designed lesson plans modeled on the problem-solving approach. The PD program took place over a period of eight months, and comprised 10 sessions.
The teachers observed filmed pedagogical situations and analyzed those films together. The common analysis during the PD led to significant discussions and reflective processes on their teaching and on explorative instruction. As part of the intervention, teachers were exposed to tasks with high cognitive demand and they were asked to teach these tasks to their students.

Teachers who participated in the study agreed that their lessons would be filmed (before, during, and after the intervention). Lesson observations lasted an entire class period, while the researcher observed, took field notes, and collected relevant classroom artifacts (i.e., instructional tasks or handouts).

These lessons were then transcribed and analyzed according to the class discourse (Cazden, 2010) and according to Schoenfeld’s (2014) description of agency. This paper focuses on the findings from the post-intervention videos.

The analysis included four main phases:

a. Analyzing the transcription of the lesson, and locating the main sections in which the teachers asked the students to justify their answers and encouraged class discussions as opposed to a discourse characterized by IRE. Sections characterized by IRE were highlighted in one color, while sections in which discussions were held vis-à-vis justifications were highlighted differently.

b. Analysis of the cognitive requirements in the lesson and the support that the teacher provided students when addressing the task.

c. Analyzing the agency of the students during the lesson.

d. Examination of the link between classroom discourse, cognitive requirements (support for learners) and students’ agency during the lesson.

The analysis was carried out by two content experts separately. After each of them conducted their separate analysis, they compared analyses and discussed the findings. In cases of disagreement, a discussion was held in order to reach an agreement between the researchers.

**Findings**

The findings summarize the analysis of the 11 lessons (one lesson per teacher) and are presented in Table 1. The summary of the findings in Table 1 presents a comparison between the lessons and exposes the link between the class discourse, cognitive demand and the student’s agency during lessons.

A number of issues emerge from Table 1: Teachers 1-7 were very supportive of the students, mainly by dividing the task into smaller parts. Moreover, the class discourse of these teachers was characterized by discussions in which the students were partially required to provide justifications and a great deal of use was made of IRE discourse. In contrast, teachers 8-11 supported students by asking questions that promoted the discussion by requiring justification and leading to conceptual connections, and in such cases students were active throughout the lesson.
The findings indicate that the use of tasks with high cognitive demands did not necessarily lead to a discourse that requires students to justify, although in all of the lessons there were some justifications. The demand for justification was expressed more significantly when the support of the students was characterized by questions asking for justification. Moreover, when the teacher divided the task into smaller parts and lowered the cognitive requirements of the task (teachers 1-7), the discourse became characterized by IRE (Cazden, 2001; 2010).

Another link was found between the support given to students by the teacher and students’ agency during the lesson. It was found that when the teacher supported the students by dividing the task into smaller tasks (whether in the plenum or in the group), the students were passive in their learning, and their agency decreased. On the other hand, when the teacher supported the students by asking questions that promoted discussion (e.g., “how do you explain what was said?”) students’ agency remained high.

Discussion

The aim of this study was to examine the link between classroom discourse, cognitive demands (and support given to learners), and student agency.

The main outcome emerging from the findings focuses on the tendency of some of the teachers to reduce the cognitive demands by dividing the tasks into sub-
tasks. These findings are consistent with previous findings (Heyd-Metzuyanim et al., 2018; Hill, Ball and Schilling, 2008; Shabtay and Levenson, in press). Researchers consider that this is due to the multi-dimensional and fragile nature of teachers’ learning processes (Santagata et al., 2011; Spillane and Zeuli, 1999).

Although the teachers taught tasks with high cognitive demands as they learned in the intervention, they actually supported the students. This support is characterized by the division of the task into smaller parts, and in fact leads to a reduction in the cognitive demands and a decrease in student agency. Previous studies (Henningsen and Stein, 1997) suggest that teachers need to know how to ascribe tasks with high cognitive demands.

This study found that when the teacher mediated the task, the mediation led to reduction of the cognitive demands, restricting the classroom discourse to one characterized by the IRE pattern, and as a result, reduced the students’ agency.

The main limitation of the study lies in the fact that it revealed the teaching in only 11 lessons. However, the findings reveal complex processes of change, and it is possible that an ongoing process with the teachers is required in order to demonstrate the characteristics of explorative instruction. It also appears that for the teachers who participated in the study, the concept of ‘explorative instruction’ is a concept that needs to be elaborated on specific teaching characteristics and practices. These concepts were discussed through a deliberate process during the intervention, especially in discussions that arose at real time and after viewing the videoed pedagogical situations.

The theoretical implication of the link between students’ agency and the nature of the classroom discussion leads to paying close attention to the construction of discussions.

Although this study is only the beginning of a journey in the learning processes of mathematical teaching, there are a number of practical implications that can be derived from it.

The first concerns the time and resources needed to support teachers in transition from traditional teaching to explorative instruction. Programs of 30-60 hours do not provide the time and resources necessary to adopt explorative instruction (Gresalfi and Cobb, 2011). Another conclusion concerns our need as teachers/educators to be patient with the process required and to allow teachers to learn from the process itself, to experiment, and to create change at their own pace and with proper support.

References


THE ROLE OF METAPHOR IN GEOMETRICAL INQUIRY – OPPORTUNITIES OFFERED BY ABSTRACT ART

Jana Slezáková, Paola Vighi and Darina Jirotková

Abstract

Recent researches on the development of geometrical thinking stress the role of communication, of different languages and systems of representation as novel and important elements in education. This paper focuses on recognition of geometrical shapes and on the development of geometrical and metaphorical languages in 10-11 year old pupils. The scenario selected is an artistic context: a Kandinsky painting, where the painter employs geometrical shapes, becomes an instrument for analysis of pupil’s description of the figures and their mutual positions. The analysis of 120 individual interviews stresses the fundamental role of metaphors in geometrical learning. Moreover, the results show the need of working also with shapes without axes of symmetry, in primary school.

Keywords: communication, recognition of geometrical shapes, geometrical thinking, painting

Introduction

Interactions between mathematical and artistic experience provide broad space for exploration of spatial thinking, of children’s preconceptions and their geometrical knowledge. The presented paper exploits an “abstract art” painting by Kandinsky (1866-1944), entitled “Soft hard” (Figure 1). This painting presents an arrangement of geometrical shapes on canvas, following artistic and aesthetic
criteria. The pleasant colours used and the apparent simplicity of the painting can motivate pupils to observe and describe it. Nevertheless, according to Arnheim (1984, p. 143), the images do not explain themselves. Thus, it is necessary to study and plan such activities that allow observation of the paintings and of the images that are reproduced.

Duval (2016, p. 214) stresses that

“The problem of the visual recognition of the shapes begins when there is not only one shape, but at least two that can be separated, overlapping, juxtaposed, partially overlapping, or one inscribed in another”.

In the Kandinsky painting (Figure 2), we can find all these kinds of reciprocal positions of shapes: for instance, shapes n. 12 and n. 19 are separated, n. 13 and n. 14 are juxtaposed, n. 20 is overlapped to n. 19, n. 2 and n. 3 are partially superimposed, n. 4 is inscribed in n. 3.

We can examine the painting from different points of view. An Euclidean examination (Figure 2) shows the representation of twenty geometrical figures: one square (6), two rectangles (1, 15), a third rectangle can be seen on the right – it is obtained by connecting two right-angled triangles (13, 14), two circles (3, 4), twelve triangles (2, 5, 7, 8, 9, 12, 13, 14, 16, 17, 18, 19), 3 “moons” (word suggested by children) (10, 11, 20). The square (6) has two sides approximately parallel to the hypotenuse of the ‘big’ red triangle (12); it is the only polygon with sides not parallel to the sides of the canvas. On the contrary, all the rectangles have the sides ‘horizontal’ and ‘vertical’ with respect to the background. There are two equilateral triangles (7, 18), five isosceles triangles (2, 5, 8, 9, 19) and six right-angled triangles (12, 13, 14, 16, 17, 19). Observing the painting through the lens of geometry of transformations, we find three symmetrical compositions of shapes: on the left, the axis of symmetry is the straight line containing the ‘axes of symmetry’ of triangles (2) and (5); the three triangles (7), (8) and (9) are arranged as a ‘symmetrical tree’; downward on the right, a ‘moon’ (20) is superimposed on an isosceles triangle (19) in a symmetrical way. Topological concepts of ‘inside or outside’ are involved and “top and down” or “forward or backward”, “on the left or on the right” and so on.
**Theoretical framework**

Close relationships between Art and Mathematics can furnish a very useful topic for geometric instruction in Primary School. Recent researches on geometrical thinking regard the role of spatial knowledge, which consists of seeing and observing objects, images, relations among them and possible transformations of one into another, as crucially important (Clements and Battista, 1992).

Abstract paintings with their composition of shapes and colours can provoke a way of seeing the world through mathematical eyes. Following Sfard (2009, p. 161)

“Mathematics starts where the tangible objects of the real life finish and where begins the reflection on our discourse about these objects”.

This is the main reason of our choice of implementing a research study that starts from a painting.

A working group on geometrical thinking organized during the CERME 8 conference identified four competencies in geometrical thinking: visual, operational, figural and reasoning (Maschietto *et al.*, 2013). Obviously, each of these competencies has its own properties but all of them are connected. The work of reading a painting that the present research involves engages all these four competencies, as the following sentences illustrate. Firstly, the ‘visual pole’ is involved, since children start observing and inspecting the painting. However, if we suggest to make a copy of the painting using some tools, we provoke a shift to the ‘operational pole’, in which also gestures play an important role. Moreover, interpretation of a drawing involves the ‘figural pole’, since it relates a representation and the represented object using properties, definitions and so on. Finally, the description and explanation of a painting promotes “reasoning in act” and argumentation (reasoning pole).

Paintings created according to the rules of the so-called “Abstract art” present shapes without a link to the real world, although we can sometimes discern some real objects in them. Duval (1993) stresses that also mathematical objects are not directly accessible, they can be known only by their representations. Duval (2005) affirms that the figures presented at school are “perceptively remarkable and culturally familiar”, but their main feature is

“to not be iconic, that not looks like an object seen and known in the reality” (Duval, 2005, p. 3).

Thus, it is necessary to promote transition from shapes to geometrical figures, from “iconic” to “not-iconic representation”, a sequence of operations that allows identification of geometrical properties.

In particular, Duval (2016) studies the cognitive act of “seeing in art and in geometry” and highlights the fundamental distinction between two different kinds of visualizations: “visual recognition of a shape” and “cognitive recognition of
the object represented from this shape”. He shows that the two activities of ‘seeing a painting’ or ‘seeing geometrical figures and constructions’ require two different kinds of visualization. He writes also that

“Any iconic recognition, also the figural representations, implies a discursive recognition, named silent verbalization, which identify the cognitive recognition of this that it shows. The spontaneous oral verbalization clarifies and amplifies this silent verbalization, without modifying it. The geometry requires the cognitive recognition with natural language, …” (Duval, 2018, p. 226).

In Van Hiele’s theory (1986), informal language appears at the first level, named the “visual level”, in which pupils recognise figures as whole, without observing their properties. In the second level, the “descriptive level”, pupils are able to recognise the figures and their properties, while the relations among properties or among different figures appear at the “rational level”.

Diffusion of communication by the mass media has recently lead researchers to study the role of different languages and systems of representation as new important aspects in education. We decided to use an artefact (a painting) to help the development of the process of communication. Interviews based on this visual mediator help us to know geometrical ideas of the interviewed pupils, studying if they use words that label geometrical objects or if they prefer metaphors or gestures. It allows investigation of how they distinguish between ‘horizontal’, ‘vertical’ or ‘slanting’ directions, and if they pay attention to symmetrical constructions, parallelism, orientation of shapes, different possible shapes of triangles.

Many researches highlight the importance of metaphors in teachers’ and pupils’ discourses and their role in didactics of mathematics. A metaphor appears as a cognitive instrument that creates the meaning of an object instead of representing it (Sfard, 1997; Lakoff and Jonshon, 1980). For us, a metaphor is a verbal, written, graphic or gestural expression that allows communication with other subjects. Jirotková (2011) distinguishes eight stages of development of language in geometry. In particular, her research focuses on the second stage of “verbal commands accompanying manipulation activity” that “transfers the pupil’s knowledge in action to knowledge in word” and on the third stage of “metaphoric language” that

“develops the ability to make metaphors, which in consequence develops the ability to uncover interrelations (Gardner, 1999, p. 305)” (Jirotková, 2011, p. 175).

The research presented in this paper asks the following questions:

- What language and, in particular, metaphors do children use in communication about geometry?
- How do children describe local relations among shapes or mutual positions between blocks of shapes?
and the main question is:

- What can we learn about a pupil’s geometrical understanding, observing their behaviour in an artistic context?

**Methodology**

We worked with 10-11 year old pupils. We conducted individual interviews based on the Kandinsky painting reproduced in Figure 1. We interviewed 120 pupils, the majority in Italy, but also in the Czech Republic (31 pupils), in ordinary classes of traditional primary schools. The researcher or the teacher made the interviews outside the classroom. Each interview lasted 10-15 minutes. It was organized as follows.

We presented a copy of the Kandinsky painting in A4 size asking the pupil to comment on it. Subsequently, we showed a copy (Figure 3) of the painting made by another pupil, Mario, explaining that he was 6 years old, i.e. younger than the interviewed pupil.

The pupils were then asked the following two questions, both of which are strictly connected to our two first research questions:

1) “What can you see in this Kandinsky painting? What can you say about it?”

2) “Please, compare Kandinsky’s painting and Mario’s composition and comment on them. Can you find any differences? Explain them and justify.”

When the interview was coming to an end, we posed other questions related to recognition of geometrical figures: How many rectangles are there in the painting? How many circles? How many triangles?

The study should support the transition from a synthetic to an analytical way of seeing in geometry. The employment of the Kandinsky painting could appear too difficult because of the number of shapes and of the different kinds of them. However, in our opinion, its richness and complexity can promote real geometrical understanding.
Let us start with some comments on Mario’s composition (Figure 3). The part on the left is satisfactory, although the triangle 2 is rotated: in fact, it is isosceles, but it seems equilateral, since there is very little difference between the lengths of its sides. In the right part it is possible to find more arguable dispositions of the shapes, the most evident is that many of them are rotated (i.e. their sides are not parallel to the sides of the canvas, as in the original painting). In fact, the left part is simpler and easier to copy than the other part. Moreover, in Mario’s composition, some pieces partially exit from the sheet of paper that represents the background (canvas), maybe because of manipulation.

Other observations: circle 4 is over triangles 2 and 5 (in the original painting it is not clear, but it seems to be under the triangles); triangle 9 is not in the right position with respect to 8 and 7; the block composed by 7, 8, 9 is too far from the block 2, 3, 4, 5; parallelism between the sides of the triangle 12 and 19 is not respected; triangles 16 and 17 are upside down; 13 do not touch 12; 14 is posed in an incorrect way; 13 and 14 do not form a rectangle; 18 is turned; 20 does not respect the line of symmetry of 19.

Results

Related to the comments on Kandinsky painting (Figure 1), we noticed that 10-11 year old pupils gave fewer descriptions of the painting than younger children (5-6 years old) (see Kaslová and Vighi, 2019, in print). Their initial descriptions showed little imagination. The majority said that “in the painting there are geometrical shapes” and they name them ‘circles’, ‘triangles’, ‘rectangles’, ‘a square’ or ‘rhombus’ (shape 6), ‘half circles’ and so on. Some pupils (but not many) preferred to make a description using natural language and metaphors related to real life objects. These two different behaviours could be interpreted as a consequence of the didactical contract (Brousseau, 1980). In other words, it is possible that older pupils preferred the use of geometrical language, since their schoolteacher in the last year of primary school level “wanted” from them a scholastic language. The youngest pupils might have preferred the use of natural language even if also in kindergarten they learnt the names of the main geometrical shapes. The natural language allowed them to give a description freely, without restrictions. They did not have to worry about making mistakes. Only after the teacher asked the pupils to describe the painting again, the pupils tried to add other comments, using words or metaphors related to real life objects.

In particular, the background was recognized as the “sea and boats with sails in the wind”, or “sea that lost its way and is going towards the lighthouse” or “mountains in blue sky with a star” (pointing at 6) or “eye that is seeing the wind”. Some pupils observing the painting from the left to the right discovered “the evil and the right” or “the sadness and happiness”, showing in this way the emotional aspects of their perception. Block 6, 7, 8 appeared as “tree that is falling”. Sometimes they were attentive to the particulars; for instance, a girl said: “in the big circle there is a grass with violets and lavender”.
In the left part of the painting, the majority saw an “eye” (2, 3, 4, 5) but also “black hole” or “space shuttle” or “lighthouse in the ocean”. The block 7, 8, 9 was perceived as “a tree” or “a Christmas tree” or “a sequence of arrows”. The shape 20 was “a moon” or “half moon” or “mouth” or “wave” 20 represents “whiskers”, the block 19, 20 was described as “sad face” (see Emoticon faces in mobile phones) or the block 12, 19, 20 as “a boat on the sea”. The block 13, 14 was a “flag”, 10, 11 “hair”, 18 “a stinger”, 15 “bench”, 6 “a chair” and so on.

As far as question 2) is concerned, the first words used were “slant” or “askant” or “not quite straight” (with the meaning of ‘not horizontal’, communicated by gestures of hands, pointing at some shapes), so what was perceived first was the ‘lack of equilibrium’ in the painting. Some pupils commented on the space management: “In the right part of the painting, some figures in part exit from the paper”; one explained that “the square is too on the top”, but the majority studied the shapes with no attention paid to the background (canvas). First, attention was paid to the biggest shapes, then to the smaller ones. Pupils focused on whether the shapes were overlapping or not: for instance, “Circle 4 must be over triangles 2 and 5”. About the ‘tree’ (7, 8, 9) they said: “The triangles must be aligned” or “the vertex of this triangle (9) must be in the middle”. This was a way of communicating their idea of a lack of symmetry in the tree. Some pupils paid attention to mutual positions: e.g., “19 is rotated, so also 20 must be rotated” or “these shapes are too close (or too far)”, referring to their distances. The idea of parallelism is mediated here by the concept of distance. Again, some pupils observed that in the bottom part of 19 “the angle is bigger”, showing a well-known misconception on the angle concept.

Special attention must be paid to the pupil’s comments on and behaviour in front of block 13, 14. It was usually at the end of describing the painting that they discovered the mistake in Mario’s arrangement of triangles 13 and 14, but they were not able to describe it well. For instance, a boy said “Mario attached the longest sides of the triangles”, focusing on the sides and not on the mutual positions of the two triangles. Another claimed: “Mario exchanged the two triangles”. A girl tried to explain saying that “the red triangle (13) must touch the biggest red triangle (12)”. In fact, they touch only in one point. When pupils tried to explain why triangle 17 is incorrect, they said, for instance, “the horizontal part is down” (the major cathetus) instead of realizing that it was turned over. So, they tried to give explanations using also geometrical names and concepts. Other pupils demonstrated a major understanding of the arrangement of triangles 13 and 14, making comments such as “the red triangle is upside down” referring to the need of turning the red triangle.

As far as the final research question about the number of geometrical shapes recognized in the picture is concerned, the findings of the study are also interesting.
The number of rectangles is “one” if pupils do not recognize 1 and 15 as rectangles (but as “strips”). In fact, these rectangles are usually not presented at schools; it is “two” if they recognize 1 and 15 as rectangles, and 13-14 as a pair of triangles; it is “three” when they indicate 1, 15 and 13-14.

The number of circles is “one” if they recognize only shape 4 as a circle, it is “two” (4 and 3), or “five” if they count all the curved shapes (3, 4, 20, 10, 11).

The number of rectangles varies from seven to twelve: seven is the number of triangles with an axis of symmetry (2, 5, 7, 8, 9, 17, 18) (missed recognition of not-reversible triangles as triangles), twelve is the number of triangles drawn. The number of rectangles stops at ten for those pupils who do not perceive composition n. 13-14 as a rectangle (we will comment on this in Conclusions).

The following table (Table 1) reports the percentages concerning the activity of shapes recognition.

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<td>27%</td>
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<td>circles</td>
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<td>triangles</td>
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Table 1: Shapes recognized in Kandinsky painting

Conclusions

In our experiment, we could observe how pupils built their ideas using and appreciating both, geometrical and metaphorical languages.

As the results show, the proposed activity involves many geometrical aspects: recognition of shapes, orientation, shapes arrangement, mutual positions of shapes, geometrical transformations, and organization of the sheet of paper space. The task allows to ‘deconstruct’ the painting, in Duval’s meaning (2016), observing the properties of the figures, their sides, vertices, axes of symmetry.

As we tried to explain above, the language used by the pupils brings interesting information on their geometrical understanding and their misconceptions. In our opinion, the description of the painting becomes interesting when pupils use metaphors, since they give information on their ‘real’ geometrical learning, which cannot be reduced to the use of correct words to name the shapes. This sentence is supported, for instance, by the pupils’ behaviour when facing ‘the problem of the two-coloured triangles’. Sometimes, indicating 13 and 14 arrangement, pupils said: “Here is a mistake. It is a kite, not a rectangle”, but they were not able to explain Mario’s mistake in depth. To help them, we sometimes suggested they should work with copies of triangles 13 and 14, cut out of paper and put the triangles together to obtain Mario’s kite first and then the original rectangle drawn in the painting. Some pupils (a minority) realized only after a lot of attempts they
would have to turn the triangle upside down, some of them (a small number) never did. In other words, we could observe a real difficulty of applying symmetry in the space (an inverse transformation) in the situation of manipulation. The same could be observed in a previous research study conducted with 5-6 year old pupils (Vighi, 2018). Thus, one result of this research study is that work with shapes without an axis of symmetry (referred to as ‘not-reversible shapes’) is very difficult for some pupils. In contrast, pupils who use metaphorical language, for instance “The red triangle must be put upside down”, show understanding of the movement needed to get from Mario’s kite to the Kandinsky’s rectangle. In our opinion, this signals a deeper geometrical understanding of this situation.

In fact, the ‘not-reversible shapes’ have no axis of symmetry and they are used less frequently in schools. Thus, we recommend that teachers organize activities with this kind of shapes, starting from manipulation with them and their observation in order to fill this gap. We also suggest teachers and pupils speak about symmetry and about the presence of axes of symmetry in figures. A possible follow-up activity in the class could be based on recognition of shapes with or without an axis of symmetry in the Kandinsky painting.

Metaphors reveal much about pupils’ geometrical ideas and the teacher can use them and develop them further. For instance, to describe the incorrect position of triangle 19 in respect to 12, a boy uses “askant”, another one “not horizontal”. This could be an occasion to speak of ‘parallelism’ and to discuss the meaning of the words ‘horizontal’ and ‘vertical’ (relative to the background and its sides).

As Table 1 shows, 31% of pupils find only ten triangles in the paintings, while the triangles are twelve for 62%. Children of the first group perceive the arrangement 13 and 14 globally as a rectangle, although it is made of two triangles of two different colours. So, it could be useful to develop transition from a global to a local analysis of this part of the painting. This observation promotes another reflection: Kandinsky draws only two shapes, two right-angled and congruent triangles and their juxtaposition shows a rectangle, but in the picture itself, there is not a rectangle. It suggests a very important theoretical aspect, the distinction between a ‘shape’ and a ‘figure’: the first is a real object, while the other is not. In our experiment, the two triangles are shapes, while the rectangle is a geometrical figure represented by the two coloured triangles. It is an example of transition from ‘the specific’ to ‘the abstract’. This could be one of the reasons why this part of the picture caused major difficulties, both in description and in copying.

This work documents how observation of a painting can become an inquiry if the activities proposed to the pupil encourage it.

Acknowledgement: Paper supported by the EU project ‘Investigation of the development (of mental schema) of geometrical pre-concepts and concepts in pupils of age 5 - 10 years, N° CZ.02.2.69/0.0/0.0/16¬_027/0008495’
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MULTIPLICATION STRATEGIES: PROGRESSIVE DEVELOPMENT AND (OR) SYSTEMATIC TEACHING

Marijana Zeljić, Milana Dabić Boričić and Olivera Đokić

Abstract

For several decades research is focused on investigation and development of multiplicative thinking and especially on the transition from additive to multiplicative way of thinking. In this paper we analyze the effects of additive approach in multiplication of single digit numbers on the development of multiplying strategies and flexibility in their choice. Results show that additive approach inhibits the development of efficient multiplying strategies and strategic flexibility. It has negative effects to further understanding of multiplication of single digit and two-digit numbers. Without diminishing the importance of intuitive strategies, this paper supports the stance that systematic teaching develops complex multiplying strategies and multiplicative thinking. In other words, different multiplying strategies and evaluation of their efficiency on specified examples should be part of mathematics curriculum when multiplication is initially learned.

Keywords: multiplication, multiplication strategies, curriculum, multiplicative thinking, Serbia

Introduction

Arithmetic operations and calculation algorithms are major part of elementary mathematics curriculum worldwide (Verschaffel et al., 2007). The algorithms consist of well-defined steps, and if steps are used strict and right, the correct result is expected (ibid). However, research show that it is difficult to achieve conceptual understanding of arithmetic operations, especially multiplying (Dabić and Milinković, 2015). In the literature, research is focused on investigation and development of multiplicative thinking and the transition from additive to multiplicative way of thinking (Baroody, 2006; Park and Nunes, 2000).

Multiplicative thinking has wider meaning than adding the equal addends, to which we also refer as repeated addition or additive approach in multiplication learning. Even if repeated addition often stays implicit model for multiplying, numerous authors emphasize that this model is incomplete and that qualitative change is needed for conceiving the multiplicative thinking (Baroody, 2006). Ideas that support the fact that students could memorize multiplication table, imply that they remember 100 unconnected facts and use them in problem solving (ibid). Baroody (2006) argues that this approach in learning has negative effect on the development of all components of mathematical abilities (conceptual knowledge, procedural fluency, strategic and productive abilities and adaptable reasoning) and points at the following limitations: 1. inefficiency (there are too many facts to memorize); 2. inadequate application (students use memorized facts
wrong and do not verify its correctness; 3. inflexibility (students do not develop flexible strategies for calculation).

**Multiplying strategies**

Numerous researchers are emphasizing the importance of the development and flexible use of multiplying strategies (Park and Nunes, 2000). Even if knowing and using different strategies is considered as key element of flexibility, mere use of different strategies in series of similar mathematical expressions or problems without judging their efficiency, is hardly the evidence for adaptivity of strategies (Verschaffel, Torbeyns Smedt, Luwel and Van Dooren, 2007). One can use different strategies on arbitrary way, and on the other hand consistent use of one strategy for series of arithmetic tasks with appropriate structure could be more adaptive than transitioning between different strategies (ibid). Blöte et al. (2000) argue that there are many studies which show that students who are familiar with calculation strategies are not always rational in strategy choice. Students mostly use strategies which they see as well known algorithm because they are more certain in success (Baranes et al., 1989), or they believe that they are expected to use it.

There is the question about learning the multiplication strategies. Do they develope intuitively or they have to be formaly introduced? Researchers show that students who use intuitive strategies could solve multiplication tasks before they get formal instructions about multiplication as operation (Mulligan and Mitchelmore, 1997). Later, with aging and widening the knowledge, they start to use more efficient strategie s (Steel and Funnell, 2001). This change is named strategic development (Siegler, 2007). There is also a stance that students should be introduced with new strategies on tasks with known semantic structure and when they already memorized some of the products (Mulligan and Mitchelmore, 1997). Downton and Sullivan (2017) emphasize that most approaches in multiplication learning are based on simple examples and strategies with the progressive introduction of more complex strategies. However, the study of the authors implies that encouraging students to deal with more complex tasks triggers the use of sophisticated strategies. The research included students from third grade (8 and 9 years old) who did not learn multiplication strategies before the experimental program. They were introduced only to the meaning of multiplication through some of the representations. The results of the research showed that students used various strategies in calculation of the product, which we will present later in the text.

On the other hand, research show that in period in which single digit multiplying is in the focus, the change in students’ strategies is the result of systematic teaching (e.g. Sherin and Fuson, 2005). Steel and Funnell (2001) emphasize that the improvement of the less efficient strategies into efficient strategies does not necessarily have to happen if special attention is not payed to their development.
Learning the different strategies is important for two basic reasons: 1) use of strategies has positive effect to learning and students’ achievement and 2) it is necessary condition for sustainability and widening the previously acquired knowledge (Liu, Ding, Gao, Zhang, 2015). These approaches also suggest that students should have opportunity to develop their own multiplying strategies.

**Terminology.** Some of the categories for multiplication strategies that we used in our research are categories from afore mentioned research by Downton and Sullivan (2017). The strategies that students used in this research are:

- Transitional counting (visualization of groups on representations, e.g. fingers or drawing),
- Double counting and Counting by multiples (counting in multiples or combination of counting in multiples and doubling),
- Doubling (using doubling and estimation, e.g. $7 \cdot 8 = 56$ because double 7 is 14, 14 and 14 are 28, and 28 and 28 are 56),
- Multiplicative calculation (using learned multiplication fact, e.g. if $5 \cdot 6 = 30$, and to calculate $6 \cdot 6$ one should add another 6 to the result) and
- Holistic thinking (use of distributive property, i.e. operating with known products in order to get value of unknown product).

For the analysis of the data we also used naming for some of the categories mentioned in Mulligan and Mitchelmore (1997). They summarized the multiplication strategies in five categories:

- Direct counting (counting without taking the consideration multiplicative structure),
- Rhythmic counting (counting which follows the structure of the problem, e.g. 1, 2; 3, 4; 5, 6),
- Skip counting (counting in multiples),
- Additive calculation (e.g. $2 + 2 = 4$, $4 + 2 = 6$) and
- Multiplicative calculation (as defined by Downton and Sullivan).

Sherin and Fuson (2005) grouped the categories in:

- Counting categories (various types of counting),
- Additive calculation (Repeated addition and Collapse groups and add),
- Pattern based (e.g. “10’s rule” or “9 finger rule”),
- Learned products (route, with no visible computation) and
- Hybrids (the combination of afore mentioned strategies).

All authors agree that multiplying strategies could be followed from less efficient to complex, rational and efficient, but there are different number and naming of the strategies. Sherin and Fuson (2005) consider Hybrid strategies as the most sophisticated, while Downton and Sullivan (2017) refer to Multiplicative calculation and Holistic thinking as the most sophisticated. Having into account
that mentioned research are directed to investigation of students’ intuitive strategies and our research is directed to effects of additive approach, we adapted some of the categories to the context of our research (in terms of their efficiency and sophistication).

**Method**

Our study deals with multiplying strategies of students in the second grade (8–9 years old). Mathematics syllabus in Serbia (Educational Gazette 10/2004, 20/2004, 1/2005, 3/2006, 15/2006 and 2/2008) states that students should master multiplication table for single digit numbers and corresponding examples of division (to automatism). Even if students formally learn associative property of multiplication and distributive property (as multiplying sum and difference) in the second grade, the recommended approach for multiplication of single digit numbers is repeated addition. The different strategies of multiplication are not explicitly introduced. Multiplication of two-digit and single digit number is introduced as application of distributive property, i.e. two-digit number is decomposed to sum of multiple tens and units (3 ∙ 15 = 3 ∙ 10 + 3 ∙ 5).

Goal of our research is to investigate if systematic introduction of multiplication of single digit numbers as repeated addition has negative effect on the development of multiplication strategies and flexibility in their choice. In this paper we are posing following questions. When students are formally taught to multiply using only repeated addition,

1. which strategies they use after?
2. do they calculate the product with more than one strategy?
3. are they flexible in strategy choice?

Before the data collection, students learned multiplication of first ten numbers as repeated addition for one month. They learned multiplication of every number separately. Than they had one month vacation break. Prior to the research students learned multiplication of single digit and two digit numbers.

The sample consists from 27 students of the second grade in one elementary school in Belgrade. The sample is convenience because teacher and students voluntarily applied for research. Reed et al. (2015) argue that interview is the most appropriate technique for investigating multiplying strategies. We used structured interview with open ended questions. Students were provided with paper and pencil if they felt need to write. Interview lasted about 10 minutes and took place in separate room in order to encourage students to express their thoughts freely and without tension.

Expressions used in interview implied the different strategies - the value of the first expression in the series could be used in calculating the next expressions in the series: 4 ∙ 3, 4 ∙ 6, 4 ∙ 12; 7 ∙ 5, 7 ∙ 4, 7 ∙ 9; 9 ∙ 10, 9 ∙ 8, 9 ∙ 9, 9 ∙ 11. The interviewer showed one expression on paper, then student computed in her/his
mind, said the result and explained the way of calculation and if the product could be calculated differently.

Results and discussion

The first research question refers to identification and the frequency of different multiplication strategies. The frequency and the percentage of the first strategy that students offered is presented in Table 1.

The strategy Learned products (Table 1) represents memorized products. Students got instructions in textbooks and classroom which aimed at memorizing the multiplication table. The strategy Counting & Additive (Table 1) represents counting categories: Rhythmic counting (RC), Skip counting (SC) and Additive categories: Repeated addition (RA) and Doubling. Many researchers consider Doubling as additive and unsophisticated strategy in the sense of progressive development (Mulligan and Mitchelmore, 1997; Sherin and Fuson, 2005; Downton and Sullivan, 2017). We consider this strategy as indicator of flexibility in the context of our research, in which students learned only repeated addition and from previous learning are familiar with different types of counting. Hence, we singled out the number of students who calculated using this strategy. At the end, Multiplicative calculation represents derived multiplicative fact as described above or a sort of Hybrid strategy described by Sherin and Fuson (2005). Written calculation represents multiplying single and two digit numbers and reflects the students’ need to write the standard algorithm – application of distributive property.

<table>
<thead>
<tr>
<th>Expression</th>
<th>4·3</th>
<th>4·6</th>
<th>4·12</th>
<th>7·5</th>
<th>7·4</th>
<th>7·9</th>
<th>9·10</th>
<th>9·8</th>
<th>9·9</th>
<th>9·11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incorrect answers</td>
<td>No. (%)</td>
<td>2 (7)</td>
<td>1 (4)</td>
<td>3 (11)</td>
<td>3 (11)</td>
<td>8 (30)</td>
<td>7 (26)</td>
<td>0 (0)</td>
<td>8 (30)</td>
<td>2 (7)</td>
</tr>
<tr>
<td>Strategy</td>
<td>No. (%)</td>
<td>5 (18)</td>
<td>4 (15)</td>
<td>1 (4)</td>
<td>7 (26)</td>
<td>6 (23)</td>
<td>7 (26)</td>
<td>17 (63)</td>
<td>13 (48)</td>
<td>19 (70)</td>
</tr>
<tr>
<td>Learned products</td>
<td>No. (%)</td>
<td>14 (52)</td>
<td>13 (35)</td>
<td>1 (4)</td>
<td>14 (52)</td>
<td>8 (30)</td>
<td>5 (19)</td>
<td>5 (19)</td>
<td>3 (11)</td>
<td>1 (3.7)</td>
</tr>
<tr>
<td>Counting &amp; Additive</td>
<td>No. (%)</td>
<td>5 (19)</td>
<td>1 (4)</td>
<td>1 (4)</td>
<td>1 (4)</td>
<td>7 (26)</td>
<td>1 (4)</td>
<td>0 (0)</td>
<td>1 (4)</td>
<td>0 (0)</td>
</tr>
<tr>
<td>Doubling</td>
<td>No. (%)</td>
<td>1 (4)</td>
<td>4 (15)</td>
<td>3 (11)</td>
<td>4 (15)</td>
<td>4 (15)</td>
<td>9 (33)</td>
<td>0 (0)</td>
<td>7 (26)</td>
<td>5 (19)</td>
</tr>
<tr>
<td>Multiplicative calculation</td>
<td>No. (%)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>20 (74)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>17 (63)</td>
<td>0 (0)</td>
</tr>
</tbody>
</table>

Table 1: Frequency and percentage of the first multiplying strategy that students offered

Results in Table 1 show that the largest number of students use strategies that were used in learning process (Counting & Additive). There is also significant
number of students who memorized values of expressions in the multiplication table (category Learned product). In the analysis of the data, we singled out the strategy of multiplication with 9, where students used pattern: when increasing the other factor with 1, number of tens are decreased by 1, and number of units increased by 1 (products with 9 are 9, 18, 27, 36, ...). Students used this strategy as mnemonics, hence we categorized this kind of answers as Learned product. The mentioned strategy is similar to 9 finger rule (Sherin and Fuson, 2005).

Small number of students used more flexible and adaptive strategies which we categorized as Doubling (mostly below 5% of students, Table 1). Strategy Multiplicative calculation was used by very small number of students (mostly below 15% of students). This result is expected since students were not introduced to them (Sherin and Fuson, 2005; Steel and Funnell, 2001).

Still, results are surprising because the examples are chosen to suggest the use of strategies Doubling and Multiplicative calculation. The first product that students calculated was $4 \cdot 3$, and then $4 \cdot 6$, and $4 \cdot 12$. Students did not show flexibility in calculating the second and the third expression, i.e. they did not notice that products are doubled. Mostly they used again Skip counting and Repeated addition. Likewise, in the series of products $7 \cdot 5$, $7 \cdot 4$, $7 \cdot 9$ they did not notice that the first product (easy for calculation with Skip counting 5, 10, 15, ...) is for 7 greater than the second, and that the third product $7 \cdot 9$ is the sum of the first two. Students who used more efficient strategies also did not use previously calculated products when calculating the new one.

As the most reliable indicator of additive understanding of multiplication we see strategies that students used while calculating the third group of products ($9 \cdot 10$, $9 \cdot 8$, $9 \cdot 9$, $9 \cdot 11$). If one multiplication factor is 10, response is rapid and without visible computation (Sherin and Fuson, 2005). Hence, we expected that students take this product in consideration when calculating the product with factor 9. Recognizing the quantitative relation is considered as more important than memorizing the products (Stein, Kinder, Silbert and Carnine, 2006), and students did not notice relation between the numbers in examples. Not one of the students made the mistake when calculating the product $9 \cdot 10$, while 8 of them wrongly calculated value of $9 \cdot 8$. They spent a lot of time in calculation of this examples because they used Skip counting and Repeated addition strategy. Mistakes that were frequent are the wrong number of addends and wrong computation of sums. When demonstrating Skip counting and Repeated addition strategies after memorizing the products, only one of the students corrected the wrongly memorized product, while the others (two students) did not notice the mistake, which means that they do not check the correctness of memorized product. Hence, the strategies are inefficient and inflexible (Baroody, 2006; Verschaffel et al., 2007).

We can base the argument that additive approach inhibits the development of multiplication strategies and conceptual understanding of multiplication on the
analysis of strategies that students used when they multiplied single digit and two-digit numbers. Students calculated product $9 \cdot 10$, and then $9 \cdot 11$. Seventeen students (63%, Table 1) said that they could not compute products without paper and pencil, and that it could not be computed differently from $9 \cdot 11 = 9 \cdot (10 + 1) = 9 \cdot 10 + 9 \cdot 1$. The use of distributive law in calculation of the product Downton and Sullivan (2017) see as Holistic thinking – the most sophisticated strategy. In the context of our research, we see the strategy as algorithmic and inflexible because students directly learned the mentioned algorithm, could not use it without writing and could not see that the product is equal to $90 + 9$, (Baranes et al., 1989; Baroody, 2006; Verschaffel et al., 2007). It is interesting that non of the students showed the second strategy when computing this product. They had firm belief that the product could be calculated only by using paper and pencil and distributive property. We see this as conformation of the stance that additive approach to multiplication does not result with transferable knowledge (Stein et al. 2006; Liu, Ding, Bing-Cheng and Zhang, 2015).

Small number of students miscalculated the products. We present strategies that students used when they miscalculated in Table 2. The largest percentage of miscalculated products (46%) was in category Rhythmic counting (RC), Skip counting (SC) and Repeated addition (RA), while 23% of students was wrong after using some of the strategies which we find Flexible in context of this research: Multiplicative calculation and Doubling. Only two students which miscalculated after using RC, SC or AC showed second strategy which was in the same category, while the rest that were using these strategies did not show that they are familiar with other strategies.

<table>
<thead>
<tr>
<th>Unanswered</th>
<th>Learned products</th>
<th>RC, SC or AC</th>
<th>Flexible</th>
<th>Written calculation</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>7</td>
<td>3</td>
<td>18</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>Percentage</td>
<td>18%</td>
<td>8%</td>
<td>46%</td>
<td>23%</td>
<td>5%</td>
</tr>
</tbody>
</table>

Table 2: The number, percentage and categories of students miscalculated products

Our second research question refers to alternative strategies that students use after they calculate the value of the product. In Table 3 we represent frequency, percentage and categorization of the students’ second strategies. We categorized strategies RC, SC, RA and Written calculation as Inflexible, and Doubling and Multiplicative operation as Flexible.

<table>
<thead>
<tr>
<th>Expression</th>
<th>4 · 3</th>
<th>4 · 6</th>
<th>4 · 12</th>
<th>7 · 5</th>
<th>7 · 4</th>
<th>7 · 9</th>
<th>9 · 10</th>
<th>9 · 8</th>
<th>9 · 9</th>
<th>9 · 11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflexible</td>
<td>No.</td>
<td>10</td>
<td>10</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>%</td>
<td>(37)</td>
<td>(37)</td>
<td>(4)</td>
<td>(19)</td>
<td>(19)</td>
<td>(15)</td>
<td>(22)</td>
<td>(22)</td>
<td>(7)</td>
</tr>
<tr>
<td>Flexible</td>
<td>No.</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>%</td>
<td>(8)</td>
<td>(8)</td>
<td>(15)</td>
<td>(11)</td>
<td>(19)</td>
<td>(4)</td>
<td>(4)</td>
<td>(4)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

424
<table>
<thead>
<tr>
<th>Total</th>
<th>No.</th>
<th>12</th>
<th>12</th>
<th>5</th>
<th>8</th>
<th>10</th>
<th>5</th>
<th>7</th>
<th>7</th>
<th>3</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td></td>
<td>(44)</td>
<td>(44)</td>
<td>(11)</td>
<td>(30)</td>
<td>(37)</td>
<td>(19)</td>
<td>(26)</td>
<td>(26)</td>
<td>(11)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

Table 3: The number, percentage and categories of students second strategies

The results in Table 3 show that small number of students are familiar with any other strategy than the first one they offered. The ones who offered second strategy once again showed inflexible strategy: when the first strategy was Repeated addition the second strategy was Skip counting or Repeated addition with changed order of the factors. When the first strategy was Skip counting, they used Repeated addition as the second.

To approach the students’ flexibility in strategy choice we analyzed internal consistency of used strategies with Cronbach alpha as measure. We analyzed 8 examples, because it was not justified to include examples $4 \cdot 12$ and $9 \cdot 11$ for which students thought that could be calculated only by using learned algorithm. Strong consistency ($\alpha = 0.824$) showed that students are consistent in using the same strategy in all eight examples. This result refers to inflexibility in students’ computation (Baranes et al., 1989; Verschaffel et al., 2007). Students were consistent without taking into consideration relation between factors. This is especially surprising for Repeated addition in which they did not consider the (big) number of addends.

The existence of efficient and complex multiplying strategies, despite additive learning approach, shows that students had intuitive strategies which they see as more efficient than strategies they have learned in school or that they had opportunities to learn these strategies aside from school. Previous research (Mulligan and Mitchelmore, 1997; Park and Nunes, 2000) showed that with aging and learning, students progressively develop strategies that are more sophisticated. On the other hand, Siegler (2007) suggests that variety of strategies that student use on the initial learning facilitate the future learning. In our research, students who were taught by additive approach chose strategies they were taught only and showed inflexibility in strategy choice.

**Conclusion**

One of the important questions is the way curriculum treats the informal knowledge about multiplication that students have before its formal introduction. How much teacher should encourage students to be creative and “invent” their own methods in calculation? Heirdsfield et al. (2007) discuss the challenges in balancing between explicitly given procedures and development of students’ own strategies. If there is too much encouragement, elaboration of the own procedure becomes more valuable than efficiency, while too little encouragement makes students dependent on one algorithm. There is also a question how much teacher should insist on students’ explanation and making his/her strategy explicit. If teacher insists hard, students are more likely to choose strategies that are easy to
articulate, but if teacher does not insist enough, he/she could not tell if a student understood the procedure. In this paper we have analyzed the students’ multiplying strategies in the context in which curriculum assumes Repeated addition as the only strategy. This context also implies that teachers do not pay attention to the development and articulation of different strategies. The results show that students tend to consistently use only strategies they were taught, which are Repeated addition or some of the counting strategies. The intuitive strategies are inhibited, and students could not benefit from their development. In addition, students in our research rarely used more than one strategy (and if they do, they once again used the inefficient one), so they do not have the opportunity to compare and judge efficiency of different strategies. Based on these results, our research supports the stance that multiplying strategies should be explicit part of mathematics curriculum. If multiplying is approached additively, without explicit teaching of efficient and flexible strategies, students would not be able to progressively develop their own (informal) strategies.

In this paper we were also dealing with categorization of multiplying strategies. The attention should be drawn to the discussion about the efficiency, flexibility and sophistication of the strategies in categorizations developed in previous research (Downton and Sullivan, 2017; Mulligan and Mitchelmore, 1997; Sherin and Fuson, 2005). These characteristics are discussed with assumption that students had the opportunity to develop strategies. The discussion changes, if students did not have this opportunity. For example, use of distributive law in multiplying two digit and single digit number in only one manner influenced students’ belief that this kind of products could be computed using only this strategy. Therefore, the strategy which is considered as flexible (Downtown and Sullivan, 2017) could be considered as inflexible. Without the opportunity to use multiplying strategies, students could develop belief that effective learning of multiplication is memorizing unconnected facts, which has negative effect on the development of many mathematical abilities (Sherin and Fuson, 2005).

Acknowledgement: This work is part of research project in the field of basic research Ministry of Education, Science and Technological Development of the Republic of Serbia: Concepts and strategies of providing quality basic education and pedagogical work (2011-present).

References
VIRTUAL MANIPULATIVES WITH CUBES FOR SUPPORTING THE LEARNING PROCESS

Katarína Žilková and Edita Partová

Abstract

This study focuses on ways to support the learning process and development of spatial imagination in children using physical and virtual manipulation with cubes and buildings from cubes, respectively. The aim of this study is to describe didactical reasons and processes of the development of educational applets which enable the exploration of mathematical language and reasoning (pp. 99–107). Prague: Charles University, Faculty of Education.


and creation of different representations of buildings from cubes. The incentive for the creation of these applets is centred around results from action research where we observed the adequacy of virtual manipulatives in terms of children’s educational needs, their abilities, and their interests. The design and implementation of educational applets reflects the stages of Educational design research. The result is five different applets that are intended for modelling virtual buildings from cubes and also for the creation of their representations (plan and three projections). The applets are intended for teachers and children in primary education and they are freely available.

**Keywords:** applets, buildings from cubes, physical models, primary math education, virtual manipulatives

**Introduction**

Virtual manipulatives provide a learning environment in mathematical education where pupils can work with specific representations of mathematical concepts. The most important advantage of virtual manipulatives is that they allow you to manipulate virtual objects similarly to physical haptic object manipulations. While the way we work with virtual manipulatives is different from the way we work with physical manipulatives, both types of manipulatives can complement each other. In mathematical education each of them provide a different type of representation for a mathematical term, and therefore both provide opportunities for discovering new mathematical properties and relationships. The project APVV-15-0378 deals with the design, implementation, and methods of using various physical and virtual manipulatives in elementary mathematics.

The main goal of the project is to optimize teaching materials in mathematics based on the analysis of current needs and the abilities of young school aged children. The reason for the necessity of optimized teaching materials is partially a reflection of actual social problems in Slovakia related to upcoming reform for inclusive education. Within the project, virtual manipulatives (applets) are created, which are focused on various mathematical concepts. One of them is a set of applets designed for manipulation with cubes and creating buildings from cubes. In the paper we will briefly discuss the part of the process of action research realized in a chosen (specific) Slovak school.

The topic of the research was ‘buildings from cubes and their representation’. In present research we used physical manipulatives (such as colour cubes, squared papers) and virtual manipulatives (Building houses¹). Research has shown that the use of both types of manipulatives supports the learning process, while alternating between activities using physical and virtual manipulatives helps pupils create ideas. After the research, the Java Building Applet was no longer functional, and that is why we decided to design and create our own applets. We developed a collection of applets, which enables interactive and dynamic virtual

¹ [http://www.fi.uu.nl/toepassingen/00249/toepassing_wisweb.en.html](http://www.fi.uu.nl/toepassingen/00249/toepassing_wisweb.en.html) (Note: Web link is not working at the moment.)
manipulation with cubes and buildings from cubes. New applets are more closed and they thematically differentiate between learning environments compared to Building Houses. We want to provide these products (applets) to the teachers and children for free use in their pedagogical practices and gain feedback about their effectivity.

Theoretical basis

Targeted development of spatial imagination requires an educational environment rich in perception. One type of activity developing spatial imagination from an early age is the creation of buildings from cubes, which is a natural and common activity for young children. More difficult tasks are based on representations and interpretations of building from cubes recorded on planes. These tasks relate to the transfer between 3D buildings from cubes and their 2D representations. Solving such tasks requires an appropriate level of abstraction and imagination that can be acquired only through a number of practical experiences of children.

Buildings from cubes and their representations are addressed in more detail in Jirotková (2010). In accordance with Jirotková’s terminology (2010) we used terms, such as physical and virtual models of buildings from cubes, base plans, portraits of buildings, and three projections. These representations were used for action research in the classroom as well as the creation of applets.

Under the term physical model, we understand specific buildings built from cubes. A virtual model of buildings from cubes is a simulation of real specifically chosen building from cubes in a virtual environment. While in physical models children use haptic manipulation with cubes, in virtual models the manipulation with cubes is realized using hardware and software tools. It is obvious that the virtual model of buildings from cubes is one of its representations. According to Jirotková’s terminology (2010) we will consider such virtual models of buildings from cubes as a portrait of buildings from cubes. To other portraits of building belong, for example, photography or other pictorial representations of building from cubes.

A further description of buildings from cubes includes a base plan that is a unique 2-dimensional representation of a building. Under the term base plan (base design) of a building we assume an overhead view of the building in which the number of cubes placed on top of each other is written. If building from cubes is displayed through the use of three orthogonal projections to each other (front view, side view, top view), this representation is known as three projections (three views, respectively). Although three projections must not uniquely identify buildings from cubes, tasks aimed at the development of abilities for interpreting and scribing the building from different views are useful.

All the models and representations mentioned above in the research were used in both the real and virtual educational environment. We aimed to demonstrate all three types of representation provided by Bruner (enactive, iconic and symbolic). Action-based activities (enactive) involved physical manipulation with cubes, and
According to Moyer-Pecknem, Bolyard and Spikell (2002), these compose iconic and symbolic representations. The interconnection of all three types of representations is important for the creation of abstract student ideas.

The importance of effective use of different representations and manipulatives in primary mathematical education is noted in many studies (Boggan, Harper and Whitmire, 2010; Moch, 2002; Young, 2006; Žilková, 2013; etc.). Upon the conclusions of Research on the Benefits of Manipulatives\(^2\) it was that for proper and accurate understanding of mathematical terms it is necessary to progress through three stages of representations: concrete stage, representational stage, abstract stage. These three stages may be understood as educational environments and languages for the formulation of mathematical terms. In the context of this study physical models of buildings from cubes will be represented by concrete stage manipulations, while virtual models of buildings, portraits, three projections create educational environment for representational stage.

Research

A. Objectives of the research

Present research was based on the educational action research paradigm. The educational action research was realized in the classroom and it aimed to gain evidence about children’ reaction to different types of educational interventions including work with physical and virtual models of buildings from cubes.

The aim of the educational action research was to observe, categorize, and conceptualize the phenomena (didactic situations) that occurred during the research in the context of the use of physical and virtual manipulatives with cubes. After the research, we used the conceptualized Design based research to develop author applets. We attempted to create a new virtual learning environment reflecting observed phenomena (categories) from action research.

In the following part we will briefly describe the process of action research and selection of the most important results in the context of support of the teaching process.

B. Research design in the classroom: Educational Action Research

*Educational action research* was realized as a combination of traditional and modern methods (e.g. use of tablets, interactive boards, or software supporting building houses). In the classroom we created an interactive teaching environment for the use of physical and virtual manipulatives.

The research sample consisted of children in the 3rd grade of a primary school in the border village Mučín. The school was chosen on the basis of multiple reasons. The school is well equipped with ICT facilities which enabled the use of virtual manipulatives. A modernized computer room with access to internet is located in

the school and in addition each classroom has a computer, multiple-function
devices, a laptop, interactive boards, data-projectors, and wi-fi connection.
Another reason for choosing the school is the particularity of the research sample
from the children’ needs point of view. The grade consisted of 9 children in total,
however, 4 of them were children with special needs. Although the mother tongue
of many of these children is Romanian or Hungarian, Slovak is still the official
language at the school.

We used different methods of data collection. The main method was intensive
direct pedagogical operation lasting one week from 30th of May to 2nd of June
2017. The process of teaching was recorded via cameras and all ethical principles
and standards were met. During each lesson, we investigated children's
preferences concerning the selected educational activities and methods used.
Children made judgements on each activity using three emoticons that expressed
the child's attitudes, moods, and emotions in reference to the relevant activity.

The research tool consisted of a pilot methodology that contained a succession of
tasks and activities focused on the development of abilities for interpreting or
noting down the buildings from cubes. Due to the restricted length of this paper,
we introduce only a selection of activities and tasks realized during research
(Table 1), whereas we used the terminology for description of buildings from
cubes according to Jirotková (2010).

Each task was solved in both environments. This means that children created
physical and also virtual models, and also noted down buildings from physical
virtual models, respectively. Our own printed models and tools for physical
models were created and used. As virtual manipulatives we used in that time
available Java applets Building houses³.

<table>
<thead>
<tr>
<th>Environment and languages</th>
<th>Task topics</th>
<th>Model</th>
</tr>
</thead>
</table>
| physical model          | Creation of buildings from cubes according to own imagination.  
                          | Creation of buildings from cubes according the real model (teacher, classmate). | ![Figure 1](physical/virtual) model and portrait |
|                         | Creation of buildings from cubes according to the photography of real building (Figure 1).  
                          | Creation of buildings from cubes according to the portrait or picture (Figure 2). | ![Figure 2] |
(physical/virtual) model and base plan

Creation of buildings from cubes according to base plan.
Creation of base plan of buildings from cubes (Figures 3a, 3b).

Table 1: The selection of activities and tasks of action research

During the process of action research we observed multiple didactic situations which were then consulted with the children’s teachers in order to increase the objectivity in data analysis. Video analysis provided additional data. While analysing the videos, we looked at the didactic situations that arose during the research related to the use of physical and virtual manipulatives. Subsequently, we divided the observed didactic situations into the following five categories.

Category No1. Representation and interpretation of buildings from cubes as a problem of transfer between 3D and 2D. Children had problems to note down top view (the base plan) of physical building during transfer from 3D representation to 2D representation (Figures 5a, 5b).

Figure 5a: Building from cubes and the problem with the base plan

Figure 5b: Pupils mistakes in creating base plan

Category No2. Representation and interpretation of top views of buildings as a problem of transfer between horizontal and vertical 2D representation. Children had problems noting down top view of buildings from cubes when one
representation was displayed on the board (vertically) and the second one in notebook (in the plane of working table, horizontally).

*Figure 6a: Building process on interactive boards*
*Figure 6b: Building process of blocks building based on pictures from interactive board*
*Figure 6c: Tablet as a tool in the horizontal representation of a plan*

**Category No3. Combination of different educational environments as a method for increasing attention.** Combination of virtual and physical manipulatives together with writing down the buildings on squared paper helped children to stay focused, even the children with special educational needs.

**Category No4. Motivational effect when alternating educational interventions within one task.** Children could independently choose the educational environment in which they wanted to work resulting in higher success rates in the solutions of children which also meant a higher degree of motivation when solving similar tasks.

**Category No5. Effectiveness of interactive and dynamic virtual educational environment in identification and building of cubes creation.** The Possibility of applets to rotate virtual building from cubes enabled children to examine different views on building (top, front, side) and to investigate changes in the views after adding or removing one of the cubes. In this way children were able to realize how to verify and correct their solutions and held discussions about correctness of solutions.

The interest of children to work with virtual manipulatives and the efficiency of applets when transferring between spatial and planar images motivated us to create our own applets. This occurred simultaneously to the realization that the Java applets Building houses would ultimately stop working due to the software platform on which they were implemented. Thus, we have decided to design virtual manipulatives to create building models from cubes.

**C. Virtual manipulatives with cubes**

The creation of applets itself reflects stages of *Design based research* (DBR), the conception of which satisfied our intentions to create quality educational virtual environments for manipulation with cubes. Therefore, this research design became a paradigm for development of interactive virtual manipulatives with cubes.
The effects shown by virtual manipulatives during action research have led to the development of author’s interactive educational applications (applets). Java applets Building houses made up the initial inspiration for the design, creation, implementation, and optimization of authors’ applets. However, our ambition was to create an environment for the generation of tasks and their solutions, so they reflect the abilities, needs, and interests of children. Iterative phases of creation of applets will not be described because of the length of the paper. We will focus only on the aim and brief description of final products which need to be verified in a further process to determine their effectivity in the development of imagination in children with regards to several perspectives.

The aim of the applets is to create an educational environment that enables children to model buildings from cubes in a virtual environment and to examine them from different views (top, front, side). We designed and implemented a collection of five ascending in difficulty, and different in type interactive applets. Each applet is focused on one type task (Table 2, Figures 7a, 7b). At the same time, each applet is able to generate tasks with different levels of difficulty.

<table>
<thead>
<tr>
<th>Task</th>
<th>Availability of applets</th>
</tr>
</thead>
<tbody>
<tr>
<td>APP1: Construct building from plan</td>
<td><a href="http://www.delmat.info/a/8b/">http://www.delmat.info/a/8b/</a></td>
</tr>
<tr>
<td>APP2: Create base plan of building</td>
<td><a href="http://www.delmat.info/a/8d/">http://www.delmat.info/a/8d/</a></td>
</tr>
<tr>
<td>APP3: Create top, front and right view</td>
<td><a href="http://www.delmat.info/a/8c/">http://www.delmat.info/a/8c/</a></td>
</tr>
<tr>
<td>APP4: Create building from three views</td>
<td><a href="http://www.delmat.info/a/8a/">http://www.delmat.info/a/8a/</a></td>
</tr>
<tr>
<td>APP5: Find and correct a mistake</td>
<td><a href="http://www.delmat.info/a/8e/">http://www.delmat.info/a/8e/</a></td>
</tr>
</tbody>
</table>

Table 2: List and availability of virtual manipulatives with cubes (Partová and Žilková, 2017a – 2017e)

Applets APP1 and APP2 create virtual educational environment for work with base plan. Applets APP3 and APP4 are focused on development of the ability to ‘see’ buildings from different views and to be able to interpret and represent it, respectively. The aim of applet APP5 is to develop the abilities of children to accurate locate a mistake in the building from cubes (recorded from three views) and to correct it as well.

Figure 7a: Interactive environment for work with base plan

Figure 7b: Interactive environment for work with views
All applets have the same visual design. Applets enable the addition or removal of a cube. The difficulty of each task is connected with the number of used cubes. Views from the front are on the vertical plane labelled with a dot. It is possible to rotate the vertical plane and so examine views. Feedback about correctness of a solution is ensured via emoticons (happy, sad). In case of an incorrect solution, it is possible to continue in a task until the task is evaluated as correct by the system. It is also possible to use the applets without the evaluation of the correctness of a solution and they provide an open teaching environment for modelling for teachers and children.

Working versions of applets were successively verified within the iterative phases by children of primary education and students of pedagogy for primary education. Based on continuous verification adjustments and optimization of applets from the software functionality, visual aspect, feedback, and user control point of view were realized. The didactic relevance of applets is supported by results of action research. At the same time, we realize that it would be useful to create suggestion of methodology for work with virtual manipulatives and to verify these new educational environments in terms of their effectiveness in the development of children’s imagination and support in the teaching process.

Conclusions

In the context of changing the needs of children and their interests, the aim of the study was to present reasons and ways of creation of products of research: virtual manipulatives with cubes. The concept of creating applets was based on the five categories identified during action research.

All APP1 – APP5 applets provide interactive manipulation and use of a different representations and interpretations of cubes. Therefore, they have the potential to provide help in solving the problem of transfer between spatial and plane representations (category 1, and category 2). Applets APP1 and APP2 are inverse activities, and their level of difficulty varies. Understanding the symbolism of a 2D representation of a structure (plan) will be reflected in the ability to build a building as planned. In this case, the pupil proceeds according to a simple instruction “put a prescribed number of cubes”. The opposite task, to record a 2D representation based on a building portrait (APP2), requires a creative approach and developed imagination. Choosing a real or virtual environment can be helpful in both activities (category 4).

APP3 and APP4 activities are also inverse to each other. They focused on recording three different views on the construction of a building based on three perspectives. In this case, the difficulty of recording is lower than the complexity of building. These activities reflect the problem of Category 2, while the ability to rotate a building can help eliminate this problem. The pupil can rotate the image of the building to see it e.g. from above and then draw the view from the screen. These activities contribute to the development of imagination and their ultimate
result is the ability to record different views based on mental manipulation and imagination. When building according to three given perspectives, the pupil can use the possibility of turning the floor plan, the choice of environment (category 4 and 5), combining work with cubes and applets (category 3).

APP5 focuses on the relationship between building and its 2D representation through observation of changes. The pupil observes how the building record changes as a result of the change in construction, reflecting category 5. Products are provided to teachers and children for free use in pedagogical practice. The results of validation of applets to date indicates the didactic potential of applets to support the learning process in mathematics. However, for a more precise specification of applets’ effectivity and where they are appropriate is necessary to realize another type of research that should be connected with examining the impact of educational applets on creation of children’s abstract geometric imaginations.

Acknowledgement: This study was supported by Slovak Research and Development Agency project no. APVV-15-0378 (OPTIMAT)

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GOOD QUESTIONS AND GOOD QUESTIONING: TASKS AND TALK IN MATHEMATICS AND SCIENCE CLASSROOMS

David Clarke, Carmel Mesiti, Man Ching Esther Chan, Jarmila Novotná, Kateřina Jančaříková and Alena Hošpesová

Abstract

Teacher-led discussion and discussion among students are two important elements of constructivist-led teaching. These discussions are organized in such a way so that pupils discover something new, formulate a hypothesis as well as confer about different solutions of the assigned task. The questions dealt with in the workshop are the so called ‘good questions’ characterized by their content-specific focus and a diversity of correct answers. They are one of the tools for creation of an environment supporting discussions among pupils and between pupils and teachers.

Keywords: communication in the classroom, good questions, discussion

Teacher-led discussion and discussion among students are two important elements of constructivist-led teaching. These discussions are organized in such a way so that pupils discover something new, formulate a hypothesis as well as confer
about different solutions of the assigned task. During a lesson, the teacher asks a number of questions, most of which are not prepared in advance. The questions we are interested in are the so-called ‘good’ questions developed originally by Sullivan and Clarke (see Sullivan and Clarke, 1988; Clarke and Sullivan, 1990; Sullivan and Lilburn, 2010). The main characteristics of ‘good’ questions include: content-specific focus, and the opportunity for answers at different levels of sophistication (Clarke, Sullivan and Spandel, 1992, p. 209).

Good questions are a specific type of open question that must meet the following criteria (Sullivan and Lilburn, 2010): a) there exist several answers that can be accepted; b) more than mere reference to known facts is required; c) discussion is provoked; c) includes motivational function in lifelong education; d) students may learn something when they answer it and/or discuss it; and, e) teachers can learn something about their pupils from the pupils’ answers.

This workshop is presented in three parts, each highlighting a distinct perspective of good questions and good questioning:

**Session 1** *Features of good questions and good questioning* will introduce participants to features of good questions and good questioning. Participants will be given sample tasks and sample student responses for reflection and discussion (Clarke, 1995; Sullivan and Clarke, 1988, 1991). The session will also cover characteristics of good questioning for orchestrating class discussion (Lobato, Clarke and Ellis 2005). This involves alternating between the teacher’s mathematics (initiation) and the students’ mathematics (elicitation).

**Session 2** *Use of good questions in the classroom and in teacher education* will focus on the comparison of teaching with predominance of ordinary or good questions. By ordinary questions we mean such questions that are closed and can be answered conclusively and test pupils’ knowledge of facts (Jančařík, Jančaříková and Novotná, 2013). For the description of lesson structure we will use Brousseau’s Theory of Didactical Situations (Brousseau, 1997). The use of good questions requires the teacher to be prepared. Therefore we will also deal with the questions: What problems do teachers encounter when posing good questions? What are the most common mistakes when posing them?

**Session 3** *Good questions in STEM* will focus on good questions in mathematics in comparison with science and environmental studies education. Are the same characteristics still relevant? Examples from the domain of elementary school will be used for illustrating differences between preparing good questions for mathematics and science and environmental education (Novotná and Jančaříková, 2018).

**Acknowledgement:** The research study reported in this paper was partly supported by the project PROGRES – Environmental research – UK 2017-2020.
MODELLING MOTION

Brian Doig ✱, Susie Groves and John Cripps-Clark

Abstract

This workshop builds on the results of a research project, Modelling Motion: Developing Mathematics Concepts through STEM activities, which was funded by the Australian Association of Mathematics Teachers and the Australian Academy of Science. The activities were developed from the work of Galileo Galilei (1564-1642) who challenged the scientific wisdom of the Age that had been established by Aristotle.

In this workshop, attendees will perform activities, using the materials now available on the world-wide web, with a view to establishing the rôle of such activities as a part of

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a STEM programme that develops students’ new mathematics, and not merely employs already known mathematics as a tool.

**Keywords:** mathematics, STEM, modelling motion, Galileo

**Introduction**

This workshop involves some activities from the complete suite of *Modelling Motion*. All necessary equipment is provided, including student workbooks. Attendees are asked to work with the materials, as would students, and to engage in post-activity discussions in both student and teacher rôles.

**Session 1** will begin with some background to the work of Galileo and the *reSolve Project*.

Our research showed that many children have no idea of speed as a derived unit of measure. The focus is on either time or distance, but rarely on both. As many of the activities in *Modelling Motion* require measurement of speed in term of time and distance, it is necessary to investigate what speed means conceptually.

To unravel what ‘speed’ is, the initial question poses the question ‘Can you walk at a constant speed, and if so, how can we tell?’

A discussion follows.

Paper streamers are used to record the distances that people travel in successive equal time intervals. A streamer is laid out next to the path of the movement and participants place markers to show the position at each time interval. The streamer can then be cut into sections and arranged as a column graph.

The graph is then interpreted through a whole group discussion. Note that the use of streamer graphs enables the mathematics to explain the physics.

**Session 2:** In this next session we will employ Galileo’s ‘gravity diffuser’ that allows us to measure how far a ball travels during successive time intervals. This was a crucial experiment that allowed Galileo to attack the Aristotelian precepts of motion.

Our activity follows that of Galileo and builds on our experiences from Session One. The mathematics, again, is fundamental to explaining the physics. Finally, we will experiment with what happens to objects that are free-falling under gravity, and thus establish a mathematical relationship between distance fallen and time.

Discussion will, of course, round off our activity.

**Session 3:** In this final session we will investigate what happens when a motion is the result of two forces. Unlike the previous activities, this activity involves participants in observing a slow motion and explaining how what they see comes about. While this is a simple experiment, the mathematics is not so simple for those of us with experience with a Cartesian plane.
The discussion at the end of this activity will recapitulate the entire set of activities and also discuss some of the other activities from the *Modelling Motion* research.

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Galileo Project: http://galileo.rice.edu/sci/theories/on_motion.html


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**CHANGING PERCEPTIONS OF MATHEMATICS WITH ELEMENTARY TEACHER EDUCATION CANDIDATES**

*Jennifer Holm*

**Abstract**

This workshop examines the phenomenon of labelling teachers as having “mathematics anxiety” and problematizes while (re)imagining different possibilities for not using this term in discussing teachers. Instead, the focus moves to thinking about the role of potentially more positive terms in describing and supporting future teachers.

**Keywords**: mathematics anxiety, preservice teachers

It is nothing new that elementary teacher education candidates bring feelings about mathematics to an education program that may inhibit mathematics learning (e.g., Alkhateeb, 2014; Boyd, Foster, Smith and Boyd, 2014). “Mathematics anxiety” has been widely used in describing the phenomenon of a mistrust or dislike for mathematics. A quick Google Scholar search shows a plethora of publications with this keyword. Wood (1988) notes that there a variety of ways that research has taken up the definition of mathematics anxiety, and from a search of the current literature, this has not changed in recent years. To highlight a couple, some authors fall on the definition of mathematics anxiety of Richardson and Suinn (1972) as “feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations” (p. 551). While other research focuses on how mathematics anxiety is more than “a feeling of dislike or worry” (Ramirez, Shaw and Maloney, 2018, p. 145). In my practice with prospective teachers, however, I have found using the term “mathematics anxiety” is both limiting and destructive.

These sessions look at the research around mathematics anxiety and the definitions found in the literature. We will also explore other terms that have

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been used in the literature to describe the feelings of mathematics that, in many ways, overlap some of the definitions used for mathematics anxiety. We will then use the lens of growth mindset (e.g., Boaler, 2016) and mathematical habits of mind, as well as mathematics content and pedagogy, to frame work in changing these conceptions of mathematics with future teachers (and their future teaching practices).

<table>
<thead>
<tr>
<th>Session</th>
<th>Aims</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Exploration of the literature around “mathematics anxiety”</td>
<td>Exploring quotes and discussion from the literature and actual comments from future teachers; Discussion on the limitations of using “mathematics anxiety” to frame discourse with teachers and teacher candidates.</td>
</tr>
<tr>
<td>2</td>
<td>Examining the stories and conceptions of future teachers</td>
<td>Presentation of work from future teachers exploring their thoughts and histories with mathematics; Discussion on what is being observed and how these ideas are potentially problematic to bring into a classroom environment as the teacher; Discussion on terminology that could potentially be more helpful or accurate in discussion of the phenomenon.</td>
</tr>
<tr>
<td>3</td>
<td>Exploring changing the perceptions of future teachers</td>
<td>Exploring mathematics tasks and activities related to mindset, mathematical habits of mind, and content; Discussion on practical issues related to changing perceptions and conceptions of mathematics.</td>
</tr>
</tbody>
</table>

References


**GRAPHICAL SOLUTION OF WORD PROBLEMS**

*Antonín Jančařík* ☏

Abstract

Solving word problems is one of the areas systematically developed from primary to upper secondary school levels. Mathematical apparatus needed for their solution is acquired by the pupils gradually. Especially on primary school level, solving of word problems is predominantly based on the use of heuristic strategies/approaches. In the workshop, one of these strategies will be presented – the use of graphical solution using area-proportional diagrams and its systematic development. The potential of this strategy will be illustrated on a set of word problems. Also, the question of the extent in which this strategy can be taught will be discussed.

**Keywords:** word problems, heuristic strategies, graphical solution, area-proportional diagrams

Introduction

Word problems are one of the critical places of mathematics and have drawn much attention of researchers. There are dozens of factors identified as having impact on pupils’ performance when solving word problems (Caldwell and Goldin, 1979). These are length of the text, its arrangement, number of data it presents to pupils (Vondrová and Novotná, 2017).

What is crucially important when solving word problems is text comprehension. In the situation when the context of the problem is close to the pupils and they understand the processes and contexts related to the wording of the problem, the success rate of their solution increases.

When working with word problems on primary school level, pupils do not have the needed mathematical apparatus – equations and their sets become available on secondary school level. Thus, they have to use other procedures in their solution – heuristic strategies (Pólya, 2004).

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**Session 1:** In the first session, results of research focusing on the use of heuristic strategies will be presented. The team lead by P. Eisenmann studied two basic aspects of the use of heuristic strategies in mathematics education. The first aspect was classification of heuristic strategies on the basis of defined properties (Přibyl and Eisenmann, 2014). Eisenmann and others showed that some strategies can be used successfully already in primary school level (Břehovský et al., 2013).

**Session 2:** In the second session, one of the strategies, graphical solution using area-proportional diagrams and its systematic development will be presented. The method is based on recording individual variables from the text of the word problem using graphs that correspond the relations in the wording of the problem by their size or proportions.

**Session 3:** In the third session, word problems of higher difficulty will be presented and the limits of the used heuristic method discussed.

**Conclusions**

The presented method of graphic way of solving word problems proved to be easy to teach. It can be applied in the solution of problems from primary to upper secondary school level. The graphical recording of data facilitates grasping of the context of problems and increases pupils’ success rate in solution. If used well it is an intermediate stage between solving using heuristic strategies and equations.

**References**


The workshop will focus on the mathematical activity of third and fourth graders to understand their conceptualization of number and numerals. These children participated in a two-year constructivist teaching-experiment (Steffe, 1980) with the collaboration of the teacher. The experiment had four working hypotheses: (1) children able to construct natural numbers as manifolds of abstract composite units would be able to understand the representation of number both in the form of numerals and in the form of number-words; (2) children who have conceptualized natural numbers as manifold of units would be able to understand the place-value notation of numbers; (3) children who have understood the place-value notation of numbers would be able to generate their own strategies to operate with natural numbers and better understand the four arithmetical operations and standardized algorithms; and (4) children who have conceptualized natural number as manifold of units would be able to conceptualize fractions as parts of wholes.

Keywords: number, number representations, numerical diagrams, fractions

Session 1: The first session will focus on the meaning of counting for constructing number as manifold of units. Children’s number sequences and mental calculations sprouted a deeper conceptualization of number. A temporary strategy was to stop pencil-and-paper calculations to concentrate on mental calculations using counting to conceptualize units (McLelland & Dewey, 1895; Steffe et al., 1983). Decreasing the amount of meaningless use of standardized algorithms was necessary to increase motivation for learning arithmetic because of the construction of relations among different units. Usually “arithmetic is taught with no specific intention of developing meanings, and the meanings which are learned are acquired incidentally and largely through the learners’ own efforts” (Brownell, 1947, p. 10). Once children started to construct relations on their own, they introduced idiosyncratic diagrams, ways of notating, and ways of speaking that made sense to them (Saenz-Ludlow, 2004). Tasks and the nature of classroom interactions will be analyzed with the audience.

Session 2: The second session will focus on the evolution of children’s arithmetical meanings. Children who were unaware of the meaning of the position of digits in numerals became aware of the meaning of the place-value notation. As a result, they came to see numerals as physical representations of manifolds of units of 10. Children’s numerical diagrams naturally emerge from their own re-conceptualizations of number as manifold of units. Decomposing numbers into different units became a collective strategy for arithmetical operations. Numerical diagrams served these children well not only to operate with numbers but also to

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verbally describe their own numerical strategies. Children’s numerical diagrams supported their continual meaning-making of the place-value representation and their understanding of standardized arithmetical algorithms. Different tasks and numerical diagrams will be analyzed with the audience. Diagrams are analyzed as conceptualizing tools using the Peircean theory of semiotics (De Wall, 2013; Stjernfelt, 2011).

**Session 3:** The third session will focus on children’s emerging conceptualization of fractions mediated by their understanding of number as manifold of units. Numerical diagrams and metaphors mediated an initial conceptualization of fraction as parts of wholes, which was quantified as multiples of unit-fractions. Different tasks will be analyzed and shared with the audience. Children’s conceptualization of fractions will be analyzed from the semiotic perspective of diagrammatic reasoning (Stjernfelt, 2011).

**References**


**TEACHING AND LEARNING GEOMETRY WITH DIGITAL TOOLS**

*Florian Schacht and Ruth Bebernik*

**Abstract**

This workshop presents and discusses learning environments for inclusive mathematics classrooms (grade 6-8, 12-14 yrs.) in which the students (with and without special educational needs) explore geometrical concepts by using dynamic geometry software (DGS). The learning environments are part of a design-based research project, which

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aim at a better understanding of learning processes in inclusive geometry lessons. In the light of empirical examples, the workshop addresses questions regarding adequate design principles, using and combining multiple representations (like DGS or Geoboards), and the role of language (in the context of using digital tools).

**Keywords**: inclusive education, learning environments, geometry, digital tools, representation modes

In this workshop, different geometrical learning environments for inclusive classrooms – designed with and without digital elements – will be presented. The learning environments will be discussed from different perspectives. On the task level, the focus is on the potential of digital tools for teaching mathematics (Duval, 2006; Drijvers, 2002). Furthermore, design principles are presented which are essential for the developed environments for inclusive classrooms (Bakker, 2018).

On the individual level, individual geometrical concepts of different students (with and without special educational needs) are observed in respect to different representation modes (Bruner, 1966; Duval, 2006). Beyond that, on the social level, interaction processes between students during collaborative learning situations (Wollring, 2016; Schacht and Bebernik, 2018) are considered. In this context, the features of reconstruction tasks are presented and discussed.

**Session 1**: Theoretical introduction of learning environments for inclusive mathematics education

- workshop introduction and overview
- theoretical concepts for inclusive classroom settings
- digital tools and their potential for mathematical explorations
- presentation of design principles for learning environments in inclusive settings
- introduction and analysis (exploration, reflection and further development) of the learning environment “Exploring Reflectional Symmetry”
- discussion of the results with regard to an inclusive mathematics classroom

**Session 2**: Individual concept formation (focus on representation modes)

- presentation of individual concepts in collaborative settings by using different tools (digital/non-digital)
- discussing the role of representations
- introduction and analysis (exploration, reflection and further development) of the learning environment “House of Quadrilaterals”
- discussion of the results with regard to individual support

**Session 3**: Features of reconstruction tasks in interaction processes

- special characteristics of reconstruction tasks in collaborative learning settings
- introduction and analysis (exploration, reflection and further development) of the learning environment “Properties of Geometric Solids”
- discussion of the results (focus: supporting interaction processes)
RESOLVE: MATHEMATICS BY INQUIRY: PROMOTING A SPIRIT BY INQUIRY IN SCHOOL MATHEMATICS

Kristen Tripet and Olive Chapman

Abstract

resolve: Mathematics by Inquiry is an Australian government funded project to enhance the mathematics engagement and achievement of students using inquiry-oriented approaches to learning. This workshop will engage participants in two central aspects of the resolve project, with a focus on the elementary grades. Participants will explore examples of the resolve teaching resources and discuss how to support teachers’ learning and use of these tasks through communities of inquiry.

Keywords: resolve, mathematics inquiry tasks, inquiry learning communities

The workshop presents the resolve: Mathematics by Inquiry project and explores its goal of promoting a spirit of inquiry in school mathematics. resolve: Mathematics by Inquiry is an Australian Government funded project designed to promote more relevant, rigorous and engaging mathematics from Foundation to Year 10. It has two specific objectives: the first is the development of a coherent suite of resources promoting mathematical inquiry; the second is the engagement of the profession through a cohort of “Champion teachers” (or Champions) educated in the nature and use of the resources.

The resolve suite of resources aims to develop and disseminate a suite of innovative, high quality mathematics teaching and learning resources for Foundation to Year 10 school students, teachers and school leaders. In this workshop, we focus on the elementary grades. The classroom teaching resources incorporate contemporary mathematics pedagogies that are aimed at transforming the teaching and learning of mathematics. A series of professional learning

References

modules focus on important elements of inquiry, highlighted by exemplary
classroom resources addressing key components of the Australian Curriculum:
Mathematics (ACARA, 2018). Engagement of the profession occurs through
a team of almost 300 Champions across all states and territories of Australia.

At the center of the project is the reSolve: Mathematics by Inquiry Protocol, which
sets out a vision for teaching and learning mathematics and underpins all aspects
of the project. The Protocol is organized around three focal points: (i) reSolve
mathematics is purposeful; (ii) reSolve tasks are inclusive and challenging; (iii)
reSolve classrooms have a knowledge-building culture

The reSolve resources aim to promote a spirit of inquiry in school mathematics.
This does not imply that mathematics should all be taught through a particular
style of ‘inquiry learning’. Instead, the resources promote high-order thinking
through mathematical reasoning and problem solving. A spirit of inquiry means
that students should ask questions, seek answers and think critically and creatively
as they employ their mathematical knowledge to solve problems.

The potential impact of any suite of resources is dependent upon how teachers
mobilize them in the classroom setting (Schoenfeld, 2006). We therefore see the
bridge between as integral to the design of the project. The reSolve Champions
are an integral part of the curriculum resources of the reSolve project. They serve
to bridge the physical resources and their enactment in the classroom.

Each reSolve Champion participated in a 12-month development program
consisting of webinars, an online platform where he/she undertakes and discusses
the reSolve professional learning modules, and trials reSolve resources with
colleagues. The key element, however, is participation in two face-to-face
workshops. The first workshop, introduced Champions to the project, to each
other and to their role within the project. The second extended over two days, with
an optional additional day focused on theory and research. This workshop
introduced the idea of professional learning communities and positioned
Champions as leaders of such communities in their contexts.

The six key elements of professional learning communities (DuFour and Eaker,
2009) with which the Champions engaged were: shared mission, vision, values
and goals; collaborative teams, focused on learning; collective inquiry; action
orientation and experimentation; commitment to continuous improvement; and
results orientation.

The Champions engaged in activities that drew on their practical knowledge to
synthesize overarching ideas about working in communities of inquiry, which
provided models for how they might work with colleagues. Through the
Champions program we hoped to build a human resource committed to inquiry
for both student and teacher learning. Their role with colleagues extends beyond
transmission or advocacy for the reSolve resources, towards building teachers’
pedagogical design capacity.
In this workshop, participants will explore a selection of the reSolve teaching tasks designed for elementary classes and examine how the resources promote a spirit of inquiry in school mathematics and exemplify the three Protocol elements. The workshop will also address the work of the Champions. It will specifically unpack the role of building teacher capacity through the development of communities of inquiry within individual schools.

References

BUILDING LEARNING OPPORTUNITIES IN CLASSROOMS OF DISADVANTAGE: RETHINKING THE LEARNING TRAJECTORIES

Jana Višňovská ✩, José Luis Cortina ✩✩ and Pamela Vale ✩✩✩

Abstract

Researching learning trajectories in classrooms of disadvantage introduces specific challenges as well as opportunities. Situating our work within design research, we illustrate the power of theoretical approaches in which close attention is paid to both learners’ conceptual developments and the means capable of supporting such developments for all learners. We illustrate how considerations of teachers’ learning can and should inform the formulation of (students’) learning trajectories, if these are to become viable outside of research studies.

Keywords: design research, learning trajectories, means of support, supporting teachers’ work, mathematics education and equity

Background

Scaling up the use of instructional innovations, especially the complex products of classroom design research, is an important research problem (Cobb, Jackson and Dunlap, 2016). It could be argued that doing so in well-resourced first-world classrooms is already difficult, and teachers in less-well-resourced settings will be even less ready to use the complex resources well. Our experiences from working in under-resourced schools and classrooms begin to sketch a different,

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much more positive perspective. Teachers, who were in many ways typical of the settings of their work, accomplished unparalleled learning for their students. We argue that two elements were important: (a) the resources designed with the teacher’s learning and use at the fore of design considerations (cf. Cobb, Zhao and Višňovská, 2008), and (b) the professional development support the teachers experienced. We suggest that we need to treat the conditions of teacher support, which make use of the products of design research locally viable, and the ways in which such conditions could become accessible, as research questions. While learning trajectories and progressions have the capacity to inform top-down systemic interventions, we discuss how such uses alone have been problematic, especially in classrooms of disadvantage.

**Session 1: On classrooms of disadvantage and what is possible.** In this session, participants will engage in reviews and analyses of materials that will illustrate (a) typical starting points for student and teacher learning in classrooms of disadvantage (specifically, in Mexican and/or South African elementary contexts); and (b) documented outcomes of the teacher’s and all students’ learning that was facilitated in these settings by the designed means. These experiences will serve as a backdrop for review and analysis of features of a *Fractions as Measures* instructional sequence, implicated in the findings.

**Session 2: On equity-driven design commitments, heuristics, and products.** Working in classrooms of disadvantage necessitated specific commitments in theoretical positioning on the part of the research team, and resulted in the formulation of learning trajectories that differ from those established in more advantaged settings. The participants’ engagement with the instructional sequence on *Fractions as Measures* (Cortina, Višňovská and Zúñiga, 2014) will be guided towards the identification of designers’ theoretical assumptions, commitments, and specific decisions. Designed means of supporting *teacher learning* will be highlighted, and discussed in relation to an overarching instructional heuristic of making learning experiences *coherent from students’ point of view*. We may explore how and why instructional sequence addresses some of the following themes, and how these themes aid in supporting students’ as well as teachers’ learning: (a) developing a classroom culture (as part of a *mathematical* learning trajectory); (b) developing students’ need for each mathematical innovation; (c) using story as a means of providing coherence and purpose to individual learning activities; (d) providing multiple and different opportunities to develop and demonstrate the emerging mathematical *reasoning* with the innovation; and (e) adopting a need-based perspective on symbolizing and language.

**Session 3: On learning trajectories and the teacher.** Many classrooms of disadvantage are faced with “one size fits all” governmental interventions, where unwavering high expectations for student learning are among the central means of improvement. Such interventions often “fail to accommodate the extreme
backlogs in learner knowledge” (Graven, 2016, p. 8) that are typical in most classrooms in these settings. However, adjusting the content to learners’ needs is positioned as ‘lowering the bar’, and thus unacceptable. What are the teachers expected to do in classrooms where systemic support resources might be of good quality, but are unsuitable for students who are yet to reach the assumed learning levels? What resources do these teachers need, and how can we support them to transition from using resources as a means for covering and assessing prescribed content to using them as a guide for supporting their students’ mathematical reasoning, while reducing the backlog? We will explore some of these issues.

References


POSTERS

USING ICT IN BILINGUAL MATHEMATICS CLASSES – AN EXAMPLE

Eileen Baschek

Abstract

The aim of this research was to identify the linguistic and technical knowledge pupils acquire in bilingual mathematics classes while using ICT. In this paper it is explained which competences bilingual classes should achieve and how bilingual learning and ICT fits. Afterwards, the process of testing and its results are described.

Keywords: primary school, bilingual classes, mathematics

Research Topic

The improved foreign language learning of pupils in bilingual classes has been proven multiple times and caused by the higher frequency of language contact. However, there still are questions left: How keen are the competences in foreign language learning and what happens with the technical learning of a subject? As the everyday use of ICT and especially the internet offers some possibilities to learn something about foreign areas in an authentic way, it seems advantageous to combine those two areas.

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**Theoretical Framework**

Coyle (1999) describes four building blocks of bilingual classes in her “4Cs Framework” which indicate the competences for bilingual learning. *Communication* means extending the pupils’ linguistic knowledge and abilities, which qualify them for an oral academic interaction in class. The aspect *Content* contains learning new knowledge and is geared to curricular guidelines of the subject. Its aim is a double and profound knowledge acquisition. According to *Culture* the pupils have to develop a reflexive attitude vis-à-vis to their own and other cultures. The aspect *Cognition* means all cognitive abilities which can be established, such as metalinguistic knowledge or strategy learning.

This research aims to think about why combined digital and bilingual learning can be successful. International networking offers options to work with authentic and multi-media material including a linguistic reality. The Internet can support bilingual classes with its diversity, however, one must consider the host of information and its reliability. A WebQuest can limit the information because it can set a focus on important aspects.

**Empirical Study**

The method WebQuest, invented by Dodge and March in 1995, is a project-oriented and web-based learning approach for using internet sources. Schreiber adapted the method for primary school children and called it PrimarWebQuest. It should make a connection between digital and analogue sources, which are chosen by the teacher in advance. The learners research on their topic and present the results. The requirements of a PrimarWebQuest should be made transparent for the learners. They have to be able to self-evaluate their learning process (Langenhahn and Schreiber, 2012). In bilingual WebQuests pupils can read every information in both languages. Pupils can choose the working language because of two different language columns. The open-ended task enables a collaborative dialogue, offering the possibility to check the pupils’ understanding of mathematical terms.

The bilingual PrimarWebQuest was tested in a fourth grade of a primary school which offers bilingual classes (German-French). The class has been split into three different topics of symmetry and each topic has been split once again into two groups. One group had to present in German, the other one in French. The language of the working sequence was chosen freely by the pupils.

**First Results and Conclusion**

The pupils’ results show a more static use of technical terminology for the group presenting in German. That group utilized everyday terminology and periphrases for describing their insights. The other group, presenting in French, showed a more proper use of technical terminology and compared both languages. It was possible to observe a subject-based discourse in both groups. Furthermore, there
have been discussions between the groups of both languages which made a contribution to the negotiation of some terms. For the group presenting in French, that language was an obstacle as well as a motivation. In the beginning, they had difficulties to get into the topic and struggled with expressing their thoughts during the sequence. However, those difficulties have been a stimulus for the pupils to work with both languages. They compared terms and discussed their meanings for understanding the topic. Those pupils expanded some strategies in foreign language learning and have shown both, a productive and receptive expansion of vocabulary.

To conclude, the pupils felt safe because of the open choice of working language. Nonetheless, that liberty did not challenge them to use both languages. As a result, the group presenting in French worked with more language awareness and had a better cognitive stay of technical terms in both languages. Seen from the mathematical perspective, both groups were able to learn the same content.

References


TYPICAL MISTAKES IN SOLVING ALGEBRAIC PROBLEMS OF GIFTED PUPILS AT PRIMARY SCHOOLS

Irena Budínová ✉

Abstract

A research of gifted fourth and fifth graders was carried out in 2012 and 2013 to determine the solving strategies of gifted pupils when dealing with algebraic word problems. Such tasks can be solved arithmetically and algebraically. The aim of the research was to find out whether the solution strategy differs for gifted and other pupils.

Keywords: algebraic word problems, gifted pupils, solving strategies

Theoretical background

Algebraic problems, i.e. tasks solvable by equations, can be solved by experiment, arithmetic strategy and algebraic strategy. The most sophisticated is an algebraic strategy that is abstract for elementary school pupils. For all strategies, mathematical notation is challenging for pupils – a data ordering in the case of experiment, an implicit notation (Booth, 1988) in the case of arithmetic solution, an incorrect use of parentheses in an algebraic strategy.

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Methodology
The aim of this research was to determine the predominant strategies of gifted pupils to solve word problems of the equational character.

This research consisted of 135 pupils from fourth and fifth grade from whom there were 32 gifted ones. Pupils attended the mathematic club and solved various mathematical problems. Solutions were evaluated for gifted and other pupils.

Conclusions
The differences between gifted and other pupils were not so evident, although gifted pupils chose more sophisticated solutions strategies (Budínová, 2018). There were three typical mistakes in pupils’ solutions: The implicit notation, incorrect use of brackets, and uncontrolled grammar of algebraic station.

Examples of pupils’ solutions
The implicit notation is illustrated in Figure 1. A gifted pupil had been solving an atypical word problem: Grandfather gives nuts to his grandchildren. If he gave them 10 peanuts to everyone, he would miss 4 nuts. If he gave 8 peanuts to everyone, 6 nuts would be missing. How many grandchildren does grandfather have and how many nuts?

Pupil in Figure 1 solved the task arithmetically, used implicit notation ($5 \cdot 10 = 50 - 4$) and gave the wright answer: Grandfather has 46 nuts and 5 grandchildren.

![Figure 1: Sample of implicit notation](image)

The problem can be solved by an equation of two unknows or by an equation $10x - 4 = 8x + 6$ where the unknown represents the number of grandsons. We can see a typical mistake – the implicit notation where $5 \cdot 10$ on the left side is not equal to 46 on the right side of the equation. Implicit way of thinking is considered for instance in Booth (1988).

Another problem that occurred was the incorrect use of brackets. Both gifted and other pupils did not use any brackets and they wrote and used the operations as listed, as can be seen in Figure 2. Let’s see this phenomenon in practice: I am thinking of a number. If I add 7 to it, then divide this sum by three and finally multiply it by five, then I will get number 45. What is the number I am thinking about?
For gifted pupils, we have been able to find algebraic solutions in some cases. In Figure 3, however, we can see a calculation without using brackets again. A gifted fifth grader created an equation and then used gradually inverse operations.

References


**EXPLORATION OF FRENCH TRAINERS’ BELIEFS ON PROBLEM SOLVING**

*Richard Cabassut* and *Arnaud Simard*

**Abstract**

We present an exploratory research based on an online questionnaire to better understand the French teacher educators’ beliefs on problem solving from grades 1 to 3 of primary school.

**Keywords**: beliefs, problem solving, in service education, France

**Aim of the research and theoretical framework**

Method of conception and analysis of the online questionnaire

124 in-service educators answered with either multiple choice questions or Lickert scale questions. The questions are inspired by the official resources produced by the Ministry of Education (MEN, 2018) and the questions and by previous research (Cabassut, 2017). In-service educators from two different regional areas received the online questionnaire and could voluntarily and anonymously answer. The statistical analysis uses SPAD software which provides frequency tables (Table 1).

<table>
<thead>
<tr>
<th>An extract of the results of the questionnaire</th>
<th>agree</th>
<th>neutral</th>
<th>disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>In the three first years of elementary school it is necessary to rely on a diagram or a table to solve a problem</td>
<td>54</td>
<td>14</td>
<td>32</td>
</tr>
<tr>
<td>Problems modelised by subtraction are more difficult than problems modelised by addition</td>
<td>32</td>
<td>21</td>
<td>47</td>
</tr>
<tr>
<td>Typical problem solving examples should be used as a systematic reference when solving problems</td>
<td>56</td>
<td>26</td>
<td>19</td>
</tr>
<tr>
<td>By using a similar schematic representation in different problems their understanding is facilitated</td>
<td>71</td>
<td>18</td>
<td>11</td>
</tr>
<tr>
<td>If no student finds the method expected to solve a problem, the teacher can then propose the expected method.</td>
<td>57</td>
<td>19</td>
<td>24</td>
</tr>
<tr>
<td>Evaluating a problem-solving in-service education is easy</td>
<td>7</td>
<td>37</td>
<td>56</td>
</tr>
<tr>
<td>I feel able to design problem-solving in-service education tasks</td>
<td>56</td>
<td>26</td>
<td>19</td>
</tr>
<tr>
<td>I do not have enough in-service education resources on problem solving</td>
<td>43</td>
<td>33</td>
<td>24</td>
</tr>
<tr>
<td>Among teachers the notion of problem is not clear</td>
<td>67</td>
<td>23</td>
<td>10</td>
</tr>
<tr>
<td>To give the good methods to solve a problem is a good way to train for problem solving</td>
<td>16</td>
<td>27</td>
<td>57</td>
</tr>
</tbody>
</table>

Table 1: A frequency table

Conclusion

We observe an heterogeneity of the beliefs. Some beliefs are opposite to official prescription. For example 14% disagree with the statement: *The meaning of the four operations should be acquired from grade 1* although it is a Ministry prescription on grade 1. Some beliefs are controversial. For example 71% agree the statement *By using a similar schematic representation in different problems their understanding is facilitated* although change of representation registers is important to understand mathematics concepts (Duval, 2006, p. 16). 19% of in-service educators feel unable to design problem-solving in-service education tasks.
and 43% do not have enough in-service education resources on problem solving. An on-going research will try to explain the reasons of this heterogeneity and the solution to the difficulties.

References


THE CHALLENGES AND OPPORTUNITIES OF TEACHING FINANCIAL EDUCATION TASKS IN ELEMENTARY MATHEMATICS: A STUDY WITH ROMANIAN TEACHERS

Daniela Căprioară, Annie Savard and Alexandre Cavalcante

Abstract

This communication presents a study conducted in Romania on Financial Education. Eighty-three teachers responded to an online questionnaire about their teaching practices, as well as their needs, in regard to Financial Education. Findings show that these teachers need resource. An analysis of their mathematics textbooks shed lights on how Financial Education is conceptualized in the tasks.

Keywords: financial education, mathematical tasks, teachers’ needs

Background of the study

This research aims to study the teaching practices of elementary school teachers teaching Financial Education (FE). In elementary school, FE refers to financial concepts, skills, knowledge and attitudes toward essential aspects of money in society (Savard, 2018). The Romanian national curriculum includes a component of financial education embedded into the Mathematics Curriculum (under the

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units dedicated to measurement along with distance, time, capacity, etc.). An optional course on FE is also offered, which explores a variety of financial concepts with a social orientation. In Romania, the challenge to financially educate students is particularly important, because the country is still adapting to a capitalist economy and most adults grew up under a communist economy.

**An online questionnaire on teachers’ needs in regard to teach FE**

Eighty-three teachers responded to an online questionnaire about their teaching practices, as well as their needs, in regard to FE. We also analyzed tasks on FE in all official mathematics textbooks used in elementary school (Grade 1 to 5). At the end of our questionnaire, we prompted the teachers to discuss general issues or themes regarding financial education in elementary school. Among the participants who reported *not* teaching the optional course on FE in their elementary classes, the majority (27.1%) mentioned the need for a mandatory curriculum. This result provides us with insights on the teachers' perceptions of both financial education and mathematics. Given that the elementary curriculum does include a component of financial education as part of mathematics, we conclude that these teachers do not see the connection proposed by curriculum documents and materials (textbooks). As a result, they do not perceive the math class as an opportunity to integrate FE, even when the subject being explored is money (a whole unit of the government-approved textbooks is dedicated to money as a form of measurement). Another 20.3% of the teachers also mentioned the need for more training with this regard. These two themes brought up by the participants shed light to the type of instruction required to improve the teaching of FE in the context of mathematics in Romania.

**Tasks in the mathematics textbooks**

We analyzed tasks about FE present in the Romanian mathematics textbooks. Based on our distinction between representational and pragmatic measurement (Hand, 2016), we aim to demonstrate their potential to explore deeper aspects of FE under the context provided by the mathematics curriculum. Table 1 presents the levels of FE in the mathematical tasks.

<table>
<thead>
<tr>
<th>Ranks</th>
<th>Levels</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Pragmatic measurement</td>
<td>The goal is to learn mathematics critically: the financial aspect has a central role and mathematics acts as a support for arguments and analyses.</td>
</tr>
<tr>
<td>2</td>
<td>Representational measurement</td>
<td>The goal is to learn mathematical concepts: the financial aspect is that money is conceptualized as an unit of measurement that has special characteristics (defined values for each unit: coins, bills)</td>
</tr>
</tbody>
</table>
The goal is learn mathematical concepts; the financial aspect does not matter, however it can create an obstacle if not known (change, profit, etc.)

<table>
<thead>
<tr>
<th>1</th>
<th>Financial contexts</th>
</tr>
</thead>
</table>

Table 1: The levels of Financial Education tasks in mathematics textbooks

If the goal of FE in elementary school is to develop students’ understanding of the essentials of money, we believe that mathematics teachers should explore the higher levels of FE tasks in a way that fosters critical thinking. Consequently, the Romanian teachers need training in regard to the proper implementation of tasks proposed by the materials approved by the government.

References


FROM TRAINING EDUCATORS TO STUDENTS LEARNING: THE “CALCULATION CHALLENGE”

Christine Chambris and Agnès Batton

Abstract

The paper presents a trainer training project for mental calculation and the beginning of an associated research project. Resource based on a challenge of calculation is developed and used to train and enrol trainers, teachers, and students.

Keywords: mental computation, trainer education, trainer training, scaling-up

Recent results in calculation of French students in national or international surveys make the issue of teacher training a vivid societal issue in France. After years with little in-service teacher-training in mathematics, ministry of education decreed, from year 2017-2018, 9 hours a year in numerical domain for all grade-4-5 (polyvalent) teachers. How can effective training be designed and carried out under these conditions? This huge question involves at least: “content” of the training, trainers (often polyvalent too, rarely with initial training in mathematics) and trainer-training modalities. At the international level, scaling up in mathematics teacher training has become a vivid question over the years: How

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can results with a small number of teachers be scaled up in a sustainable way? Roesken-Winter, Hoyles and Blömeke (2015), e.g., identify levers that help to remove obstacles that have been hindering research for decades: it characterizes training systems for scale change and reviews theoretical tools available today to study such problems. Given this context, we present the system under study.

**A three-dimension-system: an object, actors, training methods**

*The Calculation Challenge (CC hereafter), an object developed in an IREM action research group (teachers, trainers, researchers): Training before challenging*

The cornerstone of the project consists in setting up a competition of calculation (with items such as 36 x 5 x 2 = or (0,3 + 0,2) x 6 =), between at least two classes in which student is first asked to choose, for each item, the calculation means: mental calculation or calculator. Fewer points are earned with the latter. The need to train beforehand in order to develop knowledge and skills in mental calculation appears. This implies to set up students training sessions that mix different kinds of knowledge: memorization of numerical facts, properties of operations, place value knowledge (e.g., Butlen, 2007). The group has developed teacher resources: a grid with mathematical properties in relation with items, an organized list of items outlining the involved mathematical properties, a series of training sheets of 5 items each for students as an example. The two firsts are in-constant-progress. We assume that, with *ad hoc* training, teachers develop in mental calculation teaching when setting up students training (e.g., choosing or inventing items, year-planning and implementing teaching sequences involving a given property and/or family of numbers, etc.).

**The actors: The IREM group as trainer-trainers, three groups of pre-and-in-service-teacher trainers, both with various backgrounds (e.g., specialist or not in math)**

In 2017, the group presented a brief trainer-training-scenario (Chambris et al., 2018), based on CC. It convinced a senior trainer (2nd author, now PhD student) that learning issues emerge with CC and make sense in view of the competition. She joined the group, suggested to develop trainer-training in the tense context of in-service teacher training, and enrolled a first group of teacher trainers.

**Training the trainers: Sandwich courses with teacher education sessions in between**

For trainer-training-courses we design tools to develop knowledge and skills in mental calculation teaching. E.g., the grid evoked above is used to discuss math properties with trainers and teachers, analyse or build materials for training and classes. Participants also provide feedback and empirical data from teachers and classes.
General questions
The purpose of the research is both to characterize the system, and to identify qualitative and quantitative effects on trainers, teachers, and students. This raises complex methodological issues. How to characterize trainer training that is run? For teachers and trainers: Does training impact understanding of the issues of mental calculation teaching, professional practices? For students: Does CC impact learning? How to study effects of the protocol per se (choice of calculation means)? Based on literature, a theoretical framework is being developed.

Some collected data and elements of methodology
Most of trainer training sessions and some teacher training sessions are video-recorded. IREM group work was audio-recorded up to 2017. In 2018/12, an in-training-trainer suggested, in a training session for all teachers of grades 4-5 (primary school) and of grade 6 (middle school) of a district, to set up CC. Their 22 classes participated in on 2019/04/01. Working sheets were scanned. This enables to ask more specific questions, and to locally precise methodology. On teachers’ side, we plan to analyse how they have taken up CC based on their use of resource, to supplement analyses with interviews, and to go in some classes to observe how commitment continues. On students’ side, we plan to analyse working sheets in order to identify possible teacher’s effect on the class, and possible evolution of students’ skills in the next competition.

References

MAYAN NUMERALS AS A RESOURCE FOR SUPPORTING FIRST-GRADE STUDENTS’ NUMERICAL REASONING
José Luis Cortina and Claudia Zúñiga

Abstract
We describe an instructional sequence designed to support first-grade students’ understandings of numerical units composed of other units. The instructional sequence builds on the basic numerical cosmology of the Mesoamerican indigenous nations and involves using a set of manipulatives based on the Mayan numerals.

Keywords: number sense, instructional design, cultural heritage, first-grade

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In our prior work in analyzing the oral number systems of seventeen Mexican indigenous languages (Cortina, 2013), we recognized a general way of conceiving the human body as a numerical reference. In these systems, twenty is a special number: it is almost universally used as a multiplicative base. Twenty is also the number of fingers and toes in a human body. Indeed, in some languages, the word used for expressing a twenty is the same, or is closely linked, to the one used for a person (as in Bats’ilk’op) or a soul (as in Hñähñü). The numbers five, ten and fifteen are also important. They are employed as additive bases in these systems. In some languages, the words used to name them are semantically linked to the one used for the hands (as in Nahuatl).

The numerical representation we have thus identified involves four sets of five elements in each, which create a larger set of twenty elements. Not surprisingly, it is consistent with the system created by the ancient Mayas to write numbers.

As instructional designers working within the tradition of Realistic Mathematics Education, we recognize in the Mesoamerican way of organizing numbers much potential. It offers a close and very realistic context in which to support students’ reasoning about units composed of other units; namely, their own bodies. In addition, the basic numerical arrangement of four sets of five elements forming a larger set of twenty is consistent with the rekenrek, an instructional resource that has been successfully used to support young students’ numerical development.

We also consider that including the numerical cosmology of the Mesoamerican indigenous nations in instruction can be helpful in pursuing some important general learning goals that are oftentimes overlooked in mathematics education, such as helping students to become aware of mathematics as part of their direct cultural heritage, and to develop ownership and a positive identity towards this discipline (Bishop, 2002).

The Instructional Sequence

We developed a short instructional sequence in which students explore how the Mayans – one of the nations of ancient Mexico – wrote numbers. Briefly described, in a series of whole-class conversations, students are initially asked to think about how humans wrote numbers before the system we currently use was invented. Next, a set of manipulatives is introduced that includes three kinds of items: (1) rounded wooden chips about 2cm in diameter, (2) typical craft sticks about 12cm long, and (3) flat river pebbles about 5cm in their longest dimension (see Figure 1).

![Figure 1: A set of manipulatives](image)

First, the students are told that they will be investigating how the ancient Mayans wrote numbers. Then, the names and values of the three symbols they will be
using are negotiated: the wooden chips are called “fingers” and have the value of one, the craft sticks are called “hands” (or hand-feet) and have the value of five, and the pebbles are called “bodies” and have the value of twenty.

Next in the sequence, the students engage in problem solving activities in which they investigate and reason about different ways of combining the symbols to express specific numbers. As an example, the number six can be represented both as six fingers (six chips) or as one hand and one finger (one stick and one chip). Critically, students are constantly asked to refer to their own bodies when explaining their solutions. This is intended to help all students understand, for instance, that showing five fingers in one hand and one in the other can be interpreted both as showing six fingers, or as showing one hand (with five fingers) plus one more finger.

As the sequence progresses, the size of the numbers that students are asked to represent increases (up to sixty), and more possibilities for representing them become available. Students are asked to think about the combination that the Mayans would have preferred. The problem-solving activity thus evolves to finding how to write a number using the least possible amount of symbols.

The instructional sequence has so far been used by four first-grade teachers. The accounts we have received from them suggest that the sequence can be an effective resource in pursuing the learning goals we mention above.

References

FROM TRAINING EDUCATORS TO STUDENTS LEARNING: THE “CALCULATION CHALLENGE”

**Lucia Csachová**

Abstract

School geometry is one of the less popular parts of mathematics. As a reason, it is often stated that pupils are less successful in geometry because they “do not” see (what they have) and are not accurate in drawing. An interesting way to improve the relationship and “seeing” can be to explore geometric shapes in folk ornaments.

**Keywords**: folk ornament, Čičmany, school geometry

In school geometry, pupils work with points and line segments, polygons or solids. Everything must be precisely drawn, lines must be straight, straight lines

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and planes are parallel or perpendicular to each other. But pupils often ask: where we can meet it? Sometimes, school geometry seems to forget objects which are results of different types of human activity. Connection between mathematics and art is subject of many researches. Folk art represents the specific type and can be used in education, as e.g. Slovak folk ornaments in (Gécová, 2015; Gunčaga and Zentko, 2016). The aim of our research is to explore and create folk ornaments in school geometry.

**Geometric uniqueness of folk ornaments of different cultures**

Each geographical area and culture are characterized by their own specific ornaments and patterns used in architecture, folk art or clothing. Their shape, complexity and richness are often influenced by property, social status but also by religion of people from the area. A typical example is Islam forbidding the imaging of living beings and therefore Islamic art has created very characteristic geometric motifs and shapes. It was reflected e.g. in the palace complex Alhambra in Spanish Granada.

As Slovak example we can mention well-known ornaments from the village Čičmany in the north-western region of Slovakia. There are houses decorated with stylized ornaments which are unique in the Central European region (in 2013, they were listed in the Representative List of Intangible Cultural Heritage of Slovakia). They are so specific that they have become very popular and are also used to decorate clothing, things of common using (e.g. mugs and bags), promotional items or tourist souvenirs. The origin of house decoration, which began to be used around the beginning of the second half of the 18th century, was protective and preservative, but ornaments were gradually added. The decoration was created only by women (by means of clay or lime), without any pre-drawing. Despite the fact that each house is painted differently, the women were always using the same ornamental elements and their arrangement. In addition to specific shapes, these ornaments also have their own names.

**Ornaments from Čičmany as the creative geometric element**

As the ornaments and patterns from Čičmany are very popular and known, the picture of a house from Čičmany was given to 9–10-year-old pupils and they were asked to solve two tasks:

1. “write out” geometric ornaments from decoration of this house,
2. decorate the wall of another house in a similar way.

The pupils drawn most of ornaments from the picture and tried to name them (e.g. hearts or waves) in pairs. The second task caused to start a discussion: what does
mean “in a similar way”? After agreeing that the house will be supposed to look like from Čičmany, if it has similar ornaments on the outdoor walls, they created their own decoration. Pupils tried to preserve the symmetry of the individual motifs as well as the decoration of the whole house. Some pupils created ornaments similar to that of Čičmany, others focused on novelty or symmetric decoration. We appreciate as the positive and important that children enjoyed working with the picture of the wall of the decorated house very much.

Acknowledgement: The paper was prepared with the support of the grant GAPF 1/12/2018.

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TEACHER STUDENTS’ COMPETENCE ACQUISITION IN TEACHING-LEARNING-LABS

Ninja Del Piero, Kristina Hähn, Uta Häsel-Weide, Christine Kindt, Christian Rütten, Petra Scherer and Stephanie Weskamp

Abstract
Coping with heterogeneity, e. g. identifying, supporting and developing the individual potentials and skills of all students, represents an important competence of teachers. For teacher education, practical experiences of teaching heterogeneous groups, lesson planning and analysing children’s strategies and learning processes in teaching-learning-labs at universities can help to build up those competences. In the presented project, the development of teacher students’ competences is investigated after the completion of a course for didactics of mathematics that focus on coping with heterogeneity and offers practical experiences in a teaching-learning lab. The poster will present conceptual ideas for the course, the design of the evaluation instrument and first project results.

Keywords: teaching-learning-labs, pre-service teacher education, teacher students, competence acquisition, heterogeneity

Introduction
The Universities of Duisburg-Essen and of Paderborn offer teaching-learning-labs on primary level (Good Noses Mathematics, Mathe-Spürnasen; cf. Rütten and Scherer, 2015; Rütten et al., 2018 and NumberSpace, ZahlenRaum, cf. Del Piero
and Häsel-Weide, accepted). Both labs invite heterogeneous school classes for doing mathematics, and at the same time connect the visits to the teacher education programmes. Although both labs follow different concepts and focal points, similar elements within the courses for teacher students can be identified: Practical experiences with respect to teaching heterogeneous groups or individual pupils, analysing and reflecting on pupils’ strategies and learning processes, and working on learning environments.

Research Questions and Design

In the common project “Competence acquisition in teaching-learning-labs. Out-of-school – Research-oriented – Inclusive”, both labs want to provide teacher students with reflected experiences in heterogeneous classes and research their competence acquisition. In particular, the question is how teacher students evaluate their competence development by retrospective self-assessment. In detail, we tackle the following research questions: (1) Do the teacher students notice a development of their own competences after completing a course connected to the teaching-learning-lab? (2) Which course elements do the teacher students attribute to be responsible for the increase of competence?

For measuring the retrospective self-assessment, a questionnaire has been developed (including course-specific as well as anchor items) which was completed by the teacher students at the end of the semester. To answer research question 1, an open item was used (e. g. “Describe in detail what you have learned in the seminar.”). Moreover, items were designed using a four-point Likert scale (e. g. “I can observe and interpret children’s strategies for solving tasks”, anchor item 7), and the students, on the one hand, had to rate their competence ‘before the course’, and on the other hand ‘after the course’. To answer research question 2, the questionnaire asks which activities were rated to be responsible for the increase of competences (e. g. individual lesson planning, individual reflection in contrast to common reflections or input by the educator). The data is analysed with quantitative and qualitative methods.

First results

In a pilot study, the questionnaire was tested in 2018 at both universities and was tried out in a revised version in 2019. In 2019, in total 22 students attended the seminars (12 Paderborn; 10 Essen) and filled in the questionnaires. First results (based on the revised version in both seminars) show that the teacher students notice mainly an average to high increase of competence especially with respect to (a) the knowledge of relevant subject-specific terms of the learning environments and underlying concepts as well as (b) the pupils’ difficulties (anchor items 1 & 2). They attribute their increase of competence for (a) mainly to the input of the educators (76.2 %) and to the guided activities during the seminars (57.1 %). Concerning (b) they relate this increase to the practical elements of the course: own teaching (86.4 %), reflection of experience (81.8 %),
and analyses (59.1%). In total, much relevance is attached to the integration of practical elements. The importance is expressed with the open item by a teacher student: “Collecting those practice-oriented experiences, is very important, because one can focus explicitly on pupils’ solutions, strategies und difficulties. Moreover, one could imagine, how to feel standing in front of a class, and it offers the opportunity to try out before, how to deal with different situations”.

References


A PROBLEM OF CLASSIFICATION OF FIGURES

Darina Jirotková ☞ and Paola Vighi ☜

Abstract
The poster presents how 10-11 year old pupils classify ten shapes, constructed following some given criteria. The aim is to investigate pupils’ geometrical knowledge and the use of metaphorical language.

Keywords: classification, geometrical level, metaphorical language

Introduction
Primary school pupils are set a task on a worksheet which focuses on some shapes created starting from a particular trapezium. In our research report (Jirotková, Vighi and Zemanová, 2019) we present the occurrence of the misconception “the same area, the same perimeter” in pupils’ written answers to this task. The following activity in the classroom allows each child to get acquainted with a basic trapezium by constructing it from a square sheet of paper. The next task then is “Create a shape using two congruent trapezia attached side to side”. After that, pupils are asked to classify all the created shapes and to give answer to the following question: “Are there all the possible shapes constructed according to the wording of the task or not?” In this way, they discovered the following ten shapes:

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Later, working in groups, pupils are asked to construct the ten shapes using paper and scissors and to divide them into groups, ‘families’, choosing a criterion for their classification. Finally, they give names to the families and display all in some arrangement on the flipchart. The final activity involves the comparison of the posters and of different criteria of classification.

Results and conclusions

We examined 12 posters made by Czech pupils and 12 made by Italian pupils. We analysed what attributes of shapes they considered, what names they gave to certain shapes or groups of shapes (Skemp, 1976).

The majority chose an Euclidean criterion of classification, such as ‘the number of sides’, ‘the number of angles’, ‘the presence of right angles’, ‘the kind of angles’, ‘convex or not’. Sometimes the criterion was influenced by the similarity with real-life objects (for instance, ‘a boot’, ‘a house’ …), or by the previous activity (‘figures having the same perimeter’) (Battista, 2004). Sometimes the criteria used are not exclusive (‘with right angles or obtuse angles’). Some protocols highlight confusion between convex and concave, or the problem of naming and classifying angles greater than 180°.

We could observe that pupils’ performance was influenced by teaching. For instance, the distinction between concave or convex appeared more in the Czech Republic than in Italy. This may be because geometrical activity is mainly based on convex figures in Italy.

The suggested names for groups of shapes or for individual shapes enable us to diagnose the level of pupil’s understanding of geometrical shapes. The more precise the geometrical language they use, the higher the level of geometrical thinking they expressed. The use of metaphorical language points to the fact that geometrical world has not been separated from real world yet.
WORKSHEETS IN CZECH KINDERGARTENS

Michaela Kaslová

Abstract

The poster presents five categories of criteria for evaluation of worksheets developing pre-mathematical literacy on preschool level. The criteria were used for evaluation of 2000 worksheets used in the Czech Republic (2014-2018). The results of this analysis were used in creation of semi-finished worksheets that ask for a child’s graphical action on the basis of transformation of speech to a graphical communication code and consequently transformation of the graphical code into speech. The subject of coding and decoding is space-time information.

Keywords: worksheet, graphical code, space-time imagination

Introduction

Worksheets (WS) focusing on pre-mathematical literacy represent an important part of didactical materials used in the last year of pre-primary education. We are interested in the following questions: What is the quality of these WS? How can they be characterized? What kinds of WS dominate work and what kinds are missing? Is it possible to enrich work with WS in such a way that children must process space-time data as preparation for the solution of dynamic word problems? In the study, the term worksheet developing mathematical pre-literacy refers to a sheet of paper whose aim is development of the needed abilities, knowledge and skills. A child uses it to solve a task/problem through visual communication together with other communication codes. Children’s performance was subjected to dynamic qualitative analysis.

Worksheet analysis

WS for Czech kindergartens do not need any official certification of quality. That is why 2000 worksheets were analysed (2013-2018). The following criteria of the following five categories were defined (Kaslová and Dobrovolná, 2014; Kaslová,

References


2005, 2015, 2016, 2019): I. Level of setting the task on the WS: A) mathematical correctness; B) factual correctness, C) methodological correctness; D) language level. II. Nature of the task: A) mathematical focus (key words); B) anticipated solving strategies; C) expected communication of the result (graphical, manipulation, gestures, mixed, speech; D) number of possible solutions; E) role of the task (developmental, diagnostic); F) dynamics of the task (static; dynamic); G) dimension (1D, 2D, 3D, 4D). III. Character of PL according to A) its format; B) number of tasks per page; C) context of presentation (isolated; within a series of WS, graded or ungraded). IV. Child’s skills prerequisite for successful solution: A) technical difficulty; B) transformational skills (of one communication code into another); C) orientation (number of directions); D) perception (identification, comparison). V. nature of the child’s intervention into the WS: A) finished (the child transforms graphical into spoken code); B) semi-finished, which can be a) complementary (the child completes, fills in, colours in, connects etc.); b) corrective (the child evaluates the graphical record, compares and corrects by deleting, restructuring or filling in); c) semi-finished assessment WS (the child records graphically what is right and what is wrong); C) new WS with a condition (the child creates their own WS).

Semi-finished WS for coding space-time information

The analysis showed which types of WS were scarce. 10 new WS in A4 format with dynamic tasks of V. D) were created: their aim is to have the child code time-space data and later interpret their activity. Selected WS were used in the SC1 project of University of South Bohemia. These WS are always related to a story. The child records part of the story into a given “space” using lines and arrows. By transforming from a graphical to a speech code, time-space data are usually transformed to verbs and adverbs or nouns with preposition. Administration of these WS in kindergartens in 4 regions of the Czech Republic showed that more than 92% of children could cope with these new activities

Conclusion

Classification of WS proved to be efficient and helped to uncover shortcomings in WS designed for pre-mathematical education. This classification can be used in the future for development of new types of worksheets and their testing.

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THE PROFESSIONAL LEXICONS OF MATHEMATICS TEACHERS IN THE CZECH REPUBLIC AND AUSTRALIA: HOW DO THEY COMPARE?

Carmel Mesiti *, Jarmila Novotná **, David Clarke, Alena Hošpesová *** and Hilary Hollingsworth

Abstract

By comparing and contrasting the lexicons of the Australian and Czech mathematics teachers we indicate the variation possible within two Western teaching communities with different traditions, in relation to classroom practice, and different lexico-grammatical forms and features. A comparison of both content and organisational structure revealed differences in classroom phenomena named; organisation of lexical terms by each teaching community; and perceived teacher and student agency of each of the terms.

Keywords: professional language, mathematics education, international comparison

Introduction

Teams of researchers worldwide have been engaged in documenting the professional language of middle school mathematics teachers (grades 5 to 8) as part of The International Classroom Lexicon Project. This project set out to identify the vocabulary (lexical terms accompanied by descriptions and examples) that teachers use to name the phenomena in the classroom in ten communities worldwide: the Czech Republic Australia, Chile, China, Finland, France, Germany, Japan, Korea and the USA (Mesiti and Clarke, 2017).

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Professional lexicons

The process of creation and validation of the Czech lexicon revealed some difficulty for Czech teachers to identify practices within lessons with single words or short phrases that are consistently and widely used with an agreed meaning. Czech teachers tended to use everyday language rather than pedagogical terminology. This indicated a correspondence with the traditions of a Czech approach to education, influenced by its rich history (including among others the ideas of Comenius, 1907). There is some agreement amongst researchers that the English language of practice in teaching seems particularly underdeveloped (Lortie 1975; Lampert 2000; Grossman et al. 2009) and have advocated for a framework with well-defined terms (Grossman and McDonald, 2008). In this poster, we compare the empirically documented lexicon, significant products of this international project, in current use by teachers in the Czech Republic and Australia. These lexicons encrypt the values and pedagogical history of its mathematics teaching community and give an insight into the implicit theories of learning and instruction institutionalised within this professional language of mathematics teachers.

Method and findings

The Czech and Australian researchers independently reviewed the two lexicons and began by matching lexical terms by name followed by pedagogical intention: whether identical, similar or absent. This structured comparison of the two lexicons was followed by face-to-face meetings, where preliminary pairings were scrutinised. This approach allowed for critical distinctions to emerge. In comparison, the Australian lexicon (61 terms organised into 5 categories) and the Czech Republic lexicon (49 terms organised into 10 categories) offered interesting similarities and differences. The comparison highlighted different emphases regarding mathematics classroom phenomena as indicated by the number of related terms (15 identically-named terms), the number and type of organisational categories (e.g. only one category is identically named Hodnocení/Assessment); and the encrypted agency of the terms (e.g. reflecting the importance of the teacher-student relationship for Czech teachers. The comparison of the lexicons of Czech and Australian teachers provides us with the opportunity to reflect critically on our professional vocabulary. Indeed, access to lexicons from other communities reveals possibilities for practice that may extend and expand the professional repertoire of mathematics teachers around the world and usefully inform teacher reflection on their practice.

Acknowledgements: The project was established with a Discovery Grant from the Research Council of the Australian Government (ARC-DP140101361).

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CZECH PROFESSIONAL LEXICON OF MATHEMATICS TEACHERS AT THE PRIMARY LEVEL: WHAT IS SPECIFIC FOR IT?

Hana Moraová and Jarmila Novotná

Abstract

The goal of the paper is to present a study focusing on whether and what differences are there in the language used in primary and secondary mathematics classroom. The aim of the survey is to look at whether primary and lower secondary teachers of mathematics can understand each other well when communicating about what is going on in their lessons. The survey was conducted within the frame of the international Lexicon Project: Analysing pedagogical naming systems from different cultures to reconceptualise classroom practice and advance educational theory. In the here reported study the authors worked with the Czech Lexicon that had been designed for description of any general lesson of mathematics.

Keywords: classroom practices, classroom management, communication, Lexicon, stages of a lesson

Introduction

Our interactions with classroom settings, whether as learners, teachers, researchers or policy makers, are mediated by our capacity to name what we see and experience. In (Mesiti et al., 2019), the motivation, objectives, construction of national lexicons within the frame of the International Lexicon Project are described and Australian and Czech lexicons are compared.

Work on the Czech lexicon highlighted some of the difficulties of research and practical work in the area of mathematics education. The process of validation of the Czech lexicon showed how difficult it was for Czech teachers to speak of lessons in general terms. Czech teachers tend to pass many evaluating judgements and discuss the conception of the lesson and assess its quality rather than describe...
the classroom interactions in general terms. They also tend to use everyday language rather than pedagogical terminology.

**Our research**

In this poster presentation, we study the possible differences between the languages used by teachers at different levels of education and whether there may be a misunderstanding between them when they communicate about their lessons.

The Czech Lexicon (Moraová and Novotná, 2017) is an open system where new examples can be added continually. Currently the Czech lexicon consists of ten main categories:

− Stages of a lesson (the terms that allow the description of a lesson from the point of view of its phases)
− Organization of a lesson (E.g. “Teacher gives an instruction whose aim is to make pupils behave in the lesson.”)
− Teaching methods (the terms that allow description of a lesson from the point of view of teaching methods used)
− Pupils’ individual work (E.g. “Pupils work on their own. Teacher monitors their activity.”)
− Processes supporting pupils’ learning
− Assessment
− Homework
− Organization forms of instruction
− Use of didactical means
− Type of tasks

The structure of the items in the Czech lexicon is

Description of the term – Examples – Non-examples

The Czech Lexicon was originally constructed for describing lower secondary level mathematics lessons. The process of its extension to primary level has started recently. As the lexicon not only provides a list of terms, but also illustration of each term with several examples and non-examples of what is actually meant by the term and what is not, as well as illustrative video segments, the team are fully aware of the fact that the examples, non-examples and video segments need to be supplemented by examples from primary classroom. That is why we started to study primary school mathematics classroom practices and look for what illustrates the existing terms plus if there is anything extra. This was done through an analysis of video recordings from primary mathematics lessons. Preliminary results show almost no differences.

The similarity of the Czech lexicon items for primary and lower secondary levels suggests that language of description of lessons if the terms are used properly (i.e. in the sense presented in the lexicon) used by primary teachers and lower secondary mathematics teachers is very similar and should not be an obstacle to
understanding. Using the same language by teachers will undoubtedly have positive effects on making pupils’ transition between primary and lower secondary levels easier. In other words, misunderstanding is not the result of different classroom practices but of a wrong use of terms.

References


TEACHERS AND STUDENTS’ VIEWS ABOUT INTRODUCING WEBQUESTS INTO ELEMENTARY MATHEMATICS CLASSROOMS IN QATAR

Carol Murphy ☄, Abdullah Abu-Tineh, Nigel Calder and Nasser Mansour

Abstract

The poster will present Qatari elementary mathematics teachers and students’ (Grades 5 to 6) views about the introduction of WebQuests. Qualitative data were collected from individual teacher and student focus group interviews in four classrooms before and after the introduction of WebQuests. Teachers and students’ comments indicated that WebQuests helped to develop motivation, skills, and self-learning. Nevertheless, a few students in all four classes continued to express concerns about student-centred learning in mathematics.

Keywords: inquiry-based learning, teacher-student perspectives, stress factors

Introduction

Despite an aim, to promote science, technology, engineering, and mathematics (STEM) in Qatar, students are often disengaged and disinterested in mathematics and science (Said and Friesen, 2013). These attitudes may be due to teaching in Qatar that has traditionally related to a transmissive teacher-directed model (BouJaoude, 2003). Research has suggested that inquiry-based learning (IBL) using digital technology can encourage understanding and relevance, and overcome such negative attitudes (Fullan and Langworthy, 2014).

The Study

In this project, WebQuests were introduced, in conjunction with strategies to support collaborative group work, through Professional Development (PD).

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WebQuests have been seen to support IBL and student-centred learning by inspiring students to investigate and research answers to questions online (Salsovic, 2007), but their use in Qatari schools was relatively unknown. Moving toward student-centred learning requires changes in classroom practice and responsibilities for learning, and research has suggested that teachers may experience stress factors in making these changes (Dole, Bloom and Kowalske, 2014; Grant and Hill, 2006). The main research question for this project was how teachers and students perceived the use of WebQuests, and whether they would experience stress factors in moving to a student-centred approach.

The larger project was over three years, with sixteen mathematics and science teachers and their students (Grades 5 to 9), but in this poster presentation we focus on data from individual teacher and focus group (8 students per class) interview data from the four elementary mathematics classrooms (Grades 5 to 6) over the second year of the project when the main PD took place. The students in these four classes were of a similar age and cultural background in Qatari mathematics classrooms that followed the national curriculum. Each teacher carried out at least two WebQuest topics. These topics included decimals, percentages, ratios and proportion, area of shapes, perimeters, measuring angles, and converting units of mass. Example WebQuests will be included in the poster.

**Analysis and Results**

A cross-case analysis was carried out using NVivo, with codes related to affect, acquisition of learning, and disposition to a student-centred approach. Matrix queries for pre and post-interviews for both teachers and students were carried out to identify and compare concerns and perceived benefits in relation to the codes. The poster will present tables indicating these concerns and benefits.

Key findings suggested that the teachers were positive about the introduction of WebQuests in relation to student engagement. Teacher comments focused on students’ self-learning skills, use of questioning and explanations. Many students were also positive about the introduction of WebQuests. They welcomed the opportunity to research by themselves and indicated that the range of media and sources helped them to connect ideas in the mathematics topics. Even so, there remained a small number of students (2 to 3) in each focus group who referred to stress factors such as changes in responsibility of learning.

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THINKING RELATIONALLY ABOUT FRACTION MULTIPLICATION

Helena P. Osana ☞ and Emmanuelle Adrien

Abstract

Fourth-graders’ relational thinking in the context of fraction multiplication was examined. Instructional activities that targeted gaps in relational understanding supported students’ conceptual understanding of unit and composite fractions.

Keywords: fractions, relational thinking, instruction, problem solving

One way elementary teachers can promote connected and flexible knowledge of fractions is by fostering relational thinking (Carpenter et al., 2005), a form of reasoning that prepares students for algebra in high school. We describe a teaching experiment with fourth graders as they solved word problems involving fraction multiplication. We identified gaps in students’ relational understanding and intervened with targeted instructional activities designed to help them move beyond their conceptual impasses. Our aim is to describe the ways our instructional supports promoted students’ relational thinking during problem solving.

Theoretical Framework

Relational thinking is defined as the process of relating one expression to another with the use of arithmetical properties rather than computation (Kindrat and Osana, 2018). For instance, instead of using computation to solve $\frac{1}{2} + \frac{3}{4}$, a child can think of $\frac{3}{4}$ as $\frac{1}{2} + \frac{1}{4}$, and then use the associative property to think of the problem $\frac{1}{2} + (\frac{1}{2} + \frac{1}{4})$ as $(\frac{1}{2} + \frac{1}{2}) + \frac{1}{4} = 1\frac{1}{4}$. As students solve problems with appropriate instructional support, their relational thinking shifts from being additive to multiplicative in nature (Empson et al., 2005), which forms the genesis of thinking more generally about fractions in multiplicative ways (Simon et al., 2018).

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We focused on students’ reasoning as they solved multiple groups problems, which are word problems involving a whole number of groups of a fractional amount (e.g., \( \frac{1}{2} \) cup of frosting on each of 8 cupcakes). Empson and Levi (2011) referred to two relational understandings that underpin meaningful strategies for such problems: (a) thinking relationally about unit fractions, characterized by connecting the process of partitioning \((1 \div n)\) to the size of each partition \((1/n)\), and (b) thinking about fractions as composites, an understanding that all fractions are multiples of unit fractions – i.e., relating \(m\) groups of \(1/n\) to \(m/n\).

Method

The participants were 15 fourth-graders from two classrooms in a suburban elementary school in Canada. In individual sessions, students were given three scenarios, each of which began with an equal-sharing word problem (e.g., 3 objects, 6 sharers) followed by a related multiple groups problem (e.g., 8 groups of \(\frac{1}{2}\)). Using prompts and questions, the researcher identified points in the problem-solving process where the students’ relational thinking was challenged. She responded to the students’ difficulties with activities that targeted one or more of the relational understandings described above. Activity 1 required students to partition shapes presented on sheets of paper into equal parts, which could be named and counted. Activity 2 presented students with different wholes (shapes on paper) partitioned into \(n\) parts with one part shaded. This activity emphasized the coordination of a partition \((1 \div n)\) with its size \((1/n)\). In Activity 3, the students cut out the partitions of pre-partitioned shapes so they could physically group \(m\) partitions of size \(1/n\) into composite fractions \((m/n)\).

Results

Here we describe our work with two students, Pam and Tim. In the first scenario, Pam correctly modeled and solved the equal sharing problem (6 people, 3 cookies) and accurately labeled the solution as “\(\frac{1}{2}\)” . On the multiple groups problem (8 people x \(\frac{1}{2}\) cookie), however, she drew one cookie, split it into 8 parts, and indicated that each person would get one of the 8 pieces. Thus, Pam was able to understand “\(1 \div 2\)” in the context of partitioning, but did not relate it to the size of the part (i.e., \(\frac{1}{2}\)). As she manipulated the cut-out pieces in Activity 3, Pam asked the researcher for more “cookies” as she distributed the halves to each group, thus relating \(1 \div 2\) to the quantity \(\frac{1}{2}\). When prompted, she realized that grouping the 8 halves resulted in distributing 4 whole cookies.

Tim struggled to understand fractions as composites. When working to find how many cakes would be needed if 9 people each got \(\frac{1}{4}\) of a cake, he drew 2 squares and half of a third square. He said that one cake could feed 4 people (i.e., \(1 = 4/4\)), a second cake would feed an additional 4 people, and the ninth person could get half of the third cake. When Tim was prompted to reflect on the size of the partitions, he remarked that it would “easier” if there were 5 people, showing that he had more facility seeing 5 groups of \(\frac{1}{2}\) as \(2\frac{1}{2}\) than combining 9 quarters. In Activity 3, Tim took 9 cut-out quarters and gave one to each person. By
recombining the unit partitions into wholes, he came to understand that grouping the 9 quarters resulted in “9/4”, which was equivalent to 2¼.

**Conclusion**

Our data show that short and targeted instructional activities can have a positive impact on students’ relational understanding. The activities we designed have implications for classroom practice, but the findings also support the notion that specific aspects of relational thinking can support the development of meaningful strategies for solving fractions problems.

**References**


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**AN UNUSUAL COMPARISON OF PERIMETERS**

*Ioannis Papadopoulos and Paola Vighi* 🌐

**Abstract**

Primary school pupils 10-11 years old must solve the problem of a shepherd to build a fence for his sheep. The mathematical background is the comparison of the perimeters of three figures obtained one from the other by some modifications.

**Keywords**: perimeter, area, misconception

**Aim of the study**

The aim of this study is to investigate the impact of the visual perception, and the reciprocal interaction between the concepts of ‘area’ and ‘perimeter’.

Figure 1 reproduces the three shapes presented to the students.

As it can be seen, Shape-B is obtained from Shape-A by merely moving the white rectangle, while to take Shape-C a small while rectangle must be added to Shape-
B. Perimeter is the same for all the three shapes, whereas area is the same only for Shapes-A and B, but bigger for Shape-C.

![Figure 1: The three shapes presented to students](image)

In particular, we want to test the presence of two well-known misconceptions: “Same A, same B” (Murphy, 2010) which for shapes A and B is translated to “Same area, same perimeter”, and “more A, more B” (Stavy and Tirosh, 1996) which for shapes B and C is translated to “More area, more perimeter”. Murphy (2010) examined these misconceptions in the opposite direction (from perimeter to area).

**Methodology**

A worksheet based on the shepherd’s problem was delivered to 10-11 years old students, in Greece (43 students) and in Italy (76 students), asking them to explain if the fence used for Shape-A is enough to fence also Shape-B; a similar question followed for Shape-B and Shape-C. The collected worksheets constitute our data and they were analysed on the basis of correctness and reasoning (Table 1, 2).

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Table 1: Arithmetical data for shapes A and B

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<td>12</td>
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Table 2: Arithmetical data for shapes B and C
Results and Conclusions

The analysis of the content of the worksheets confirms that visual perception hindered the fruition of the images, conditioning pupils’ answers. Tables 1 and 2 present the distribution of the student’s answers. In Table 1, the majority of the answers are correct, but with 44.07% of incorrect explanations. Many of them lie on the idea that since Shape-A and Shape-B have equal area then they necessarily have equal perimeter (same A, same B). Table 2 shows that 43.59% of the students gave wrong answers claiming that shapes B and C have different perimeters. The origin of their mistake was the belief that since shape-C has bigger area, then necessarily it will have bigger perimeter (“more A, more B”). Interestingly, a noticeable number of Italian students (26 out of 76) were able to give correct answers using correct explanation.

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USING THE REAL AND UNREAL ARTEFACTS IN DEVELOPING ALGEBRAIC THINKING

Izabela Solarz

Abstract

The aim of the classroom experiment was to answer the question of how the use of computer games and special blocks, can foster students’ difficulties with algebraic calculation. I run the experiment with the group of twenty 12-year old children, who used three different artefacts during the mathematics lessons. The results show what obstacles children can overcome, using the tools.

Keywords: manipulative and semiotic tools, algebraic symbols, difficulties

Research methodology

The tools used in the research were: Video game (DragonBox Algebra 12+, 2012-2013), computer application (Solving equations with cover-up strategy, WisWeb, 2013), algebraic blocks (Lab Gear, H. Picciotto, 1990). In course of duration of experimental teaching I collected data by observation, recording students’ arguing and collecting their written solutions. To describe the research results I analyzed difficulties that students could overcome by using the artefacts.

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Description of the classroom experiment

I used Dragon Box as a semiotic artefact by the theory of Mariotti and Bartolini Bussi (2008) Playing the game, children discovered algebraic symbols, starting from manipulating with coloured pictures on the board and finishing on solving real algebraic equations. At first they played on tablets and interactive whiteboard. Then, they continued with the solving equations as pen and pencil work, creating their own signs and giving them the mathematical meanings.

![Figure 1: Dragon Box](image)

The application “cover up” bases on doing opposite operations for solving equations. There is the difference between the equations $ax + b = c$ and $ax + b = cx + d$. You need only arithmetical operations to solve the first, but in the second ones the algebraic thinking is necessary (Filloy and Rojano, 1989). Using the application children had opportunity to investigate many equations of varying difficulty and complexity, by covering the following parts of expression and calculate their values. After working with the application they transferred the way to their writing solutions.

![Figure 2: Cover-up](image)

The Lab Gear is a complete manipulative program for teaching algebra concepts (Piciotto, 1990). The blocks are in two colours: blue and yellow. Yellow blocks represent numbers: 1, 5, 25. The blue ones represent unknowns: x, y, $y^2$, $x^2$, xy and they haven’t got the determined value. You can replace blue blocks with yellow ones, changing them to numbers. As opposed to moving around in computer environment, students can manipulate the real objects. Using the blocks, children transformed algebraic expressions: calculated their values, collected like terms, multiplied the terms, operated on sums, and also solved linear equations.
Conclusions

<table>
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<tr>
<th>Difficulties that students can overcome by using the artefacts</th>
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<tr>
<td>DB</td>
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<td>• with repeating operations on both sides</td>
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<td>• with getting used to algebraic language</td>
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<td>• with the interpretation of equal sign</td>
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<td>Cover - up</td>
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<td>• with solving more complex equations by arithmetic way</td>
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<td>• with understanding the structure of equation and connections</td>
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<td>Lab Gear</td>
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<tr>
<td>• with understanding variables, embodied by algebraic symbols</td>
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<tr>
<td>• with answer the question: how many solutions have the equation?</td>
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References


DEVELOPING LEARNING SOFTWARE FOR MIGRANT STUDENTS IN A EUROPEAN COLLABORATION PROJECT

**Andreas Ulovec**

**Abstract**

A high percentage of students in many European countries are migrants. Materials that teachers have available to teach mathematics in classrooms with a high percentage of migrant students are scarcely available, and migrant students’ learning outcomes are significantly lower than those of their native colleagues. Studies show that teachers are not sufficiently prepared to teach in multicultural contexts, and migrant students have additional challenges with learning a new language and culture at the same time as learning mathematics. In the framework of an Erasmus project, a team of educators and ICT specialists from Austria, Italy, and Slovakia is currently developing, evaluating,
testing and publishing mathematical learning software with a particular focus on migrant students. The poster will present the results from a clarification-of-terms analysis, statistical data analysis about migrant students, results from teacher interviews regarding mathematical topics where support is most urgently needed, results from the software testing in classrooms, and show some screenshots from the learning software products for elementary students.

**Keywords**: learning software, migrant students, intercultural classrooms

**Theoretical Background**

In a number of European countries, a high percentage of students are migrants (first or second generation). While most countries have succeeded to issue educational policies dealing with this situation, and suitable learning materials for languages and social sciences have recently been developed, the learning materials that teachers have available to teach mathematics in classrooms with a high percentage of migrant students lag far behind, as do the learning outcomes for migrant students (as examples, PISA 2012 for Italy shows a 32 points lower mean score for migrant students than for native students; PISA 2006 for Austria shows a 92 points lower mean score for migrant students than for native students). Several studies (Pastori, 2010; Chaloff and Queirolo Palmas, 2006) show that teachers are not sufficiently prepared to teach in multicultural contexts in which students coming from countries with different cultures and different languages have different learning styles. Learning a new language and culture at the same time as one is learning mathematics places double burden and challenges on pupils with a migrant background (Norén, 2010).

**Description of the project**

In the framework of an international collaboration project, a team of universities, schools and an ICT company, from three European countries (Austria, Italy, Slovakia) is in the process of developing mathematical learning software with a particular focus on migrant students. There is a strong need from mathematics teachers to have teaching materials available to them, that are easy to use and that provide the students with motivation to learn mathematics. To take into consideration the different learning levels and learning styles of students, it is recommended that these materials make use of ICT. It is also recommended that these materials follow a game approach instead of classical exercise-style materials. This would increase the curiosity and motivation of students, as well as decrease the anxiety and lack of self-confidence that is often prevalent in a traditional classroom setting, particularly for migrant students.

**Methodology**

The team started with a clarification of terms (immigrant students, migrant students, foreign students, students with a migrant background etc.), based on research of both scientific and educational policy literature, and with a collection of statistical data about migrant students in the partner countries. The team then
conducted interviews with teachers from elementary, upper and lower secondary schools to find out for which mathematical topics the teachers would need particular support in. These topics were developed by school-university-pairs into learning software descriptions which then have been implemented by a software company into actual learning software products. These products were tested in classrooms, evaluated by the teams’ mathematics educators and by an external evaluator using a standardized evaluation and feedback form, improved by the software company, and re-tested in classrooms. They will be published for free-of-charge use on the project website.

First results

The poster will present the results from the clarification-of-terms analysis and from the statistical data analysis, as well as the results from the teacher interviews regarding mathematical topics. It will present some results from the software testing on classrooms, and show some screenshots from the learning software products for elementary students that have been developed so far.

References


CONCEPTS FOR TEACHING UNITS IN MATHEMATICS FOR MIGRANT OR MINORITY STUDENTS

*Andreas Ulovec* and *Jarmila Novotná*

Abstract

The multicultural nature of society influences schools in many countries. Teachers are usually not sufficiently prepared to deal with a multicultural classroom context. Particularly elementary mathematics teachers feel the need for materials supporting them in teaching in multicultural classrooms. Also their pupils with a migrant background encounter more difficulties than their native classmates in acquiring basic mathematical skills. Many projects have created mathematics teaching materials in different settings, though these did not take multicultural classrooms into consideration. A very few have created concrete teaching materials for migrants, but these were rather closed teaching materials, not concepts and strategies to be further developed by teachers. A Czech-Austrian project team is currently designing concepts for teaching

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*Charles University, Czech Republic; e-mail: jarmila.novotna@pedf.cuni.cz*
units, based on the analysis of various research studies, examples of concrete teaching units based on these concepts, and guidelines on how to use these concepts. This will give mathematics teachers a tool that allows them to create their own teaching units fitting their own classroom needs. The poster will present the results from the literature analysis, examples of teaching units, and drafts of the teaching unit concepts.

**Keywords:** teaching unit concepts, migrant students, intercultural classrooms

**Theoretical Background**

The increasingly multicultural nature of modern society is one of the most significant changes to have influenced schools in many European countries, especially at elementary and middle school level. Teachers are usually not sufficiently prepared to deal with the new classroom context with pupils having a migrant or minority background, coming from countries or parts of the society with different cultures and different languages (Favilli, Stathopoulou and Gana, 2013).

Mathematics teachers of all levels, but specifically those working at elementary school levels (where the percentage of migrant students is particularly high in many countries), feel the necessity for training and materials which reflect the needs of their classes in terms of linguistic and cultural differences (Moraová, Novotná and Favilli 2015). Their pupils from minority cultures and/or those with a migrant background encounter even more difficulties than their native classmates in acquiring fundamental mathematical skills (Norén, 2008). In general, the rate of early school leavers as well as the rate of low-achieving students amongst these pupils is significantly higher than among native ones.

**Description of the project**

Based on the analysis of the results of a variety of research studies (Alrø, Skovsmose and Valero, 2010; Barwell and Kaiser, 2005; Wiest, 2002 and several others), a bilateral Czech-Austrian project team envisages the design of concepts for teaching units, and examples of the practical use of these concepts by demonstrating how to make them into concrete teaching units. This will give mathematics teachers a tool that allows them to create their own teaching units fitted to the needs of the students in their own classrooms.

**Methodology**

The project team did research on the literature for studies in multicultural and multi-ethical issues in mathematics teaching and possible educational approaches in multicultural classroom settings. We created examples of concrete teaching units specifically designed for migrant and minority students. These examples, as well as others from the literature, are currently analysed and – based on the above-mentioned research studies – draft concepts for teaching units in mathematics that allow teachers to create their own teaching units are being developed. These drafts will be evaluated, adapted and finalized. We will then develop guidelines for
mathematics teachers on how to use these concepts to create their own teaching units. We will use the results in teacher training activities, and disseminate them in the mathematics education community.

Results
The team has finished the literature analysis and creation of concrete teaching units, and is now in the process of developing the draft teaching unit concepts. The poster will present the results from the literature analysis, examples of teaching units, and close-to-final drafts of the teaching unit concepts.

References

ELIF – A CONCEPT FOR IMPLEMENTING PROBLEM-BASED LEARNING (PBL) IN 3RD AND 4TH GRADE MATHEMATICS CLASSES

Julia Wichers, Sandra Strunk and Barbara Schmidt-Thieme

Abstract
This research is focused on the development of a PBL-based teaching concept for primary school mathematics education because this student-centered teaching and learning approach is appropriate for the promotion of various competences but rare in primary education. For this purpose, a theory-based concept constitutive to PBL has been developed, tested and modified in several iterative cycles in 3rd and 4th grade mathematics classes in Germany, which concluded in an effective concept based on PBL for primary mathematics classes.

Keywords: problem based learning, learning journal, real life related learning

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Aim of the research
Caused by studies such as PISA and TIMSS, the demand for a higher application and student orientation in mathematics classes has arisen in the recent past. Problem-Based learning (PBL) as a teaching and learning approach is responsive to these requirements, while at the same time proving to be appropriate for the promotion of various subject-specific and generic competences in secondary and higher education (e.g. Dochy, Segers, van den Bossche and Gijbels, 2003; Pease and Kuhn, 2010). Corresponding concepts for mathematics classes in primary education are not yet available (e.g. Dole, Bloom and Doss, 2017). Hence our research is devoted to the question, how a teaching concept, predicated on PBL, can be designed for mathematics classes in primary schools.

Methodology and study design
The methodology of Design-Based Research (DBR) (McKenney and Reeves, 2012) is appropriate for this research question as it enables the development of a theory-based concept which can be tested in practice and developed further depending on the requirements of 3rd and 4th grade mathematics classes in Germany. First, seventeen characteristics have been generated by analyzing the general ideas of PBL considering both, diverse subject related and general pedagogical knowledge as well as results of research to design a theory-based concept constitutive to PBL.

![Figure 1: Description of the theory-based development (Strunk and Wichers, unpublished dissertation)](image)

The concept has been tested in practice and developed, employing six DBR cycles in 2018 with a duration of 7 to 10 lessons each. In each cycle, the concept has been analyzed with regard to the defined characteristics and adapted accordingly by means of the evaluative qualitative content analysis (Kuckartz, 2018). The data was collected by analyzing the children’s learning journals and an unstructured surveillance sheet documented by the teacher during and after the lessons.
Results

The result is ELIF, a concept consisting of three key components:

- Ten consecutive phases enabling self-directed learning and cooperation
- A learning journal that depicts the phases according to the children’s abilities and supports as well as structures their learning processes
- Cases as a task format which is a description of an authentic real life related situation with mathematical substance

The presentation of the authentic case leads to a central question which is mentioned by the students themselves. On that basis, the students develop individual learning objectives, usually expressed in learning questions, on which they work in groups on and subsequently reflect them according to the central question. These activities should enable the achievement of (1) skills to identify cognitive discrepancies, work self-directed on the resultant learning questions with purposive strategies and finally gain transferable knowledge, the acquisition of specific mathematical (2) process-related and (3) content-related competences.

Furthermore some principles for constructing mathematical cases for the 3rd and 4th grade as well as proceedings in the construction of a case could be outlined referring to the results of the evaluative qualitative content analysis.

References


An overview of plenary lectures from 1993 to 2019
Symposium on Elementary Mathematics Teaching SEMT
SEMT 91 – SEMT ‘19
Prague, Czech Republic, Charles University, Faculty of Education

SEMT 91
September 1 - September 5, 1991
No plenary lectures

SEMT 93
August 30 - September 3, 1993
Plenary lectures:
- Lucilla Cannizzaro: The changing faces of numbers
- František Kuřina: Constructive dimension of mathematics
- Graham H. Littler: There is more to mathematics than sums!
- Walter Szetela: North American goals and standards for school mathematics in Grades K-4

SEMT 95
August 28 - September 1, 1995
Plenary lectures:
- Lucilla Cannizzaro: On insight in problem solving activity
- Eddie Gray: Tackling the problems: An explanation for success and failure
- Milan Hejný: Development of geometrical concepts
- Graham H. Littler: Geometry for the teacher education student
- Hartwig Meissner: Geometry: Learning by doing
- Zbigniew Semadeni: Developing children’s understanding of verbal arithmetical problems

SEMT 97
August 24 - August 29, 1997
Plenary lectures:
- Ferdinando Arzarello: Assessing long term processes in the class of mathematics: The role of the teacher as a participant observer
- Sarah B. Berenson, Draga Vidakovic, Glenda S. Carter: North American perspectives on assessment
- František Kuřina, Marie Tichá, Alena Hošpesová: How to evaluate and utilise our children's geometrical experience
- Demetra Pitta, Eddie Gray: Evaluating children's mental structures in elementary arithmetic
- Dylan Wiliam: High-stakes assessment of open-ended work in mathematics: problems and possibilities
**SEMT 99**
August 22 - August 27, 1999

Plenary lectures:
- Susie Groves: Calculators, budgerigars and milk cartons: Linking school mathematics to young children’s reality
- Alena Hošpesová: Addition and subtraction concepts in school education
- Nicoletta Lanciano: Look at the Sky and the Earth through the eyes of geometry
- Erkki Pehkonen: On the use of open approach to promote an educational change in mathematics

**SEMT ‘01**
August 26 - August 31, 2001

General theme: **What is meant by the competence & confidence of people involved in the teaching of elementary mathematics?**

Plenary lectures:
- Doug Clarke: Understanding, assessing and developing young children’s mathematical thinking: The experience of the Early Numeracy research project
- Edyta Gruzczyk-Kolczynska: Why the solving of mathematical problems is so difficult for children? Intellectual and emotional conditions of mathematics education
- David Tall: What mathematics is needed by teachers of young children?
- Erich Wittmann: The O-Script/A-Script method

**SEMT ‘03**
August 24 - August 29, 2003

General theme: **Knowledge starts with Pre-conceptions**

Plenary lectures:
- John Mason: Structure of attention in the learning of mathematics
- Marie Tichá: Following the path of discovering fractions
- Pessia Tsamir, Dina Tirosh: Errors in an in-service mathematics teacher classroom: What do we know about errors in the classroom?
- Bernd Wollring: Linking pre-service and in-service teacher training: Co-operative design and Dissemination of working environments for primary mathematics

**SEMT ‘05**
August 21 - August 26, 2005

General theme: **Understanding the environment of the mathematics classroom**

Plenary lectures:
- Nadine Bednarz: A mathematics teaching approach in “weak classes”: A passage from elementary to secondary level rooted in meaning making construal
- Marja van den Heuvel-Panhuizen: Children’s perspectives of the mathematics classroom
- Bernard Sarrazy, Jarmila Novotná: Didactical contract: Theoretical frame for the analysis of phenomena of teaching mathematics
- Heinz Steinbring: Children's ways of mathematical argumentation in the classroom environment
**SEMT ‘07**
August 19 - August 24, 2007
General theme: **Approaches to teaching Mathematics at the Elementary level**
Plenary lectures:
- Kenneth Ruthven: Towards a calculator-aware number curriculum
- Petra Scherer: Investigating children learning mathematics / investigating mathematics
- Jane Watson: The development of statistical understanding at the elementary school level
- Rina Zazkis: Number theory in mathematics education: Queen and Servant

**SEMT ‘09**
August 23 - August 28, 2009
General theme: **The development of mathematical understanding**
Plenary lectures:
- Jill Cheeseman: Challenging children to think: Teacher behaviours that stimulate young children to probe their mathematical understanding
- Kath Hart: Why do we expect so much
- Sara Hershkovitz: Intuition, schema, and problem solving
- Jeremy Kilpatrick: Conceptual understanding as a strand of mathematical proficiency

**SEMT ‘11**
August 21 - August 26, 2011
General theme: **The mathematical knowledge needed for teaching in elementary schools**
Plenary lectures:
- Mariolina Bartolini-Bussi, Rita Canalini, Franca Ferri: Towards cultural analysis of content: Problems with variation in primary school
- Denise Spangler: Facilitating knowledge development and refinement in elementary school teachers
- Shlomo Vinner: What should we expect from somebody who teaches mathematics in elementary schools?
- Erich Wittmann: Early mathematical education: A plea for mathematically founded conceptions

**SEMT ‘13**
August 18 - August 23, 2013
General theme: **Tasks and tools in elementary mathematics**
Plenary lectures:
- Olive Chapman: Engaging children in learner-focused mathematical tasks
- Rose Griffiths: Working with children in public care who have difficulties in mathematics
- Günter Krauthausen: Digital media in elementary mathematics education
- Joanne Mulligan: Inspiring young children's mathematical thinking through pattern and structure
- Jennifer Young-Loveridge, Brenda Bicknell: Using multiplication and division tasks to support young children’s part-whole thinking in mathematics
**SEMT ’15**
August 16 - August 21, 2015
General theme: *Developing mathematical language and reasoning*

Plenary lectures:
- Toshiaki Fujii: Designing and adapting tasks in the Japanese lesson study: focusing on the role of the quasi-variable
- Markus Nührenbörger: Mathematical argumentation processes of children between calculation and conversion
- Irit Peled: Modelling tasks: What develops?
- Louise Poirier, Nathalie Bisaillon: The role of language and communication in the learning of maths in primary school: some examples

**SEMT ’17**
August 20 - August 25, 2017
General theme: *Equity and diversity in elementary mathematics education*

Plenary lectures:
- Herbert Ginsburg, Colleen Uscianowski: Stories, stories, and more math stories
- Mellony Graven: Blending elementary mathematics education research with development for equity – an ethical imperative enabling qualitatively richer work
- Tomáš Janík: From content to meaning: Semantics of teaching in the tradition of Bildung-centred didactics
- Esther Levenson: Promoting mathematical creativity in heterogeneous classes

**SEMT ’19**
August 18 - August 22, 2019
General theme: *Opportunities in learning and teaching elementary mathematics*

Plenary lectures:
- Berinderjeet Kaur: Opportunity to learn: Reason and communicate
- Annie Savard: How teaching mathematics in elementary school can support financial literacy education
- Ron Tzur: Elementary conceptual progressions (trajectories): Reality check + implications
- Hamsa Venkatakrishnan: Improving early number learning in contexts of disadvantage
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SEMT ‘19 was organised in cooperation with Klub přátel didaktiky matematiky na Univerzitě Karlově (Club of Friends of Didactics of Mathematics at Charles University)

Univerzita Karlova, Pedagogická fakulta
Katedra matematiky a didaktiky matematiky

International Symposium
Elementary Mathematics Teaching SEMT ‘19
Proceedings
Editors: Jarmila Novotná and Hana Moraová

Grafická úprava: Jarmila Novotná a Hana Moraová
Tisk: Retida, spol. s r.o., Dělnická 54, 170 00 Praha 7

1. vydání